An Investigation of Quantum Entanglement in Open Systems

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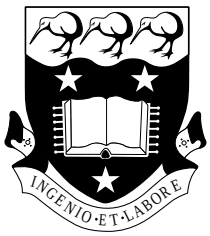

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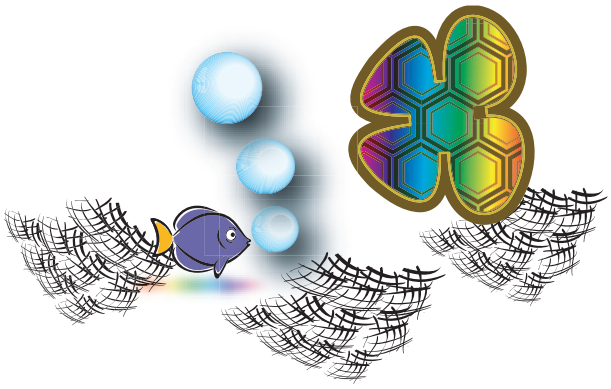

Forever lost in Hilbert Space....

Abstract

One of the primary complications in the characterisation of entanglement in realistic systems where there is active interaction with the environment is the inevitable introduction of classical uncertainty. This uncertainty introduces ambiguity in systems of interest which complicates any attempt to formulate an exact measure of entanglement within such systems. In the first part of this thesis, we demonstrate that this ambiguity is an intrinsic feature of the system resulting from its entanglement with the outside environment. An alternative measure of entanglement that actively takes the environment, and there nullifying the ambiguity, is outlined.

The advantages of this approach are demonstrated via explicit application to the cascaded qubit system. We show analytically that measurements made within the environment lead to a previously un-observed phenomenon of Quantum Cycles; stochastic trajectories that oscillate between two perfectly entangled Bell States.

In the second part of the thesis, we apply this approach to the area of entanglement engineering, where we explore methods to entangle the collective spin state of two atomic systems of variable dimensions. The work done is primarily analytical, with numerical analysis provided to further generalize or verify analytical results. Two proposals for entanglement generation are made, for limiting cases of low and high dimensional entanglement respectively.

Acknowledgments

This thesis would not have had the pleasure of existence in the Hilbert space of reality, if not for the direct and indirect contributions of the many people around me. In writing this acknowledgement, I am faced with the insurmountable task of recalling every such person, so please do forgive me if I accidentally left someone out.

First and foremost, I would like to thank my supervisors, whose time and dedication have guided me through the complex and baffling world of quantum mechanics. To Scott Parkins, whose door is always open; without someone who is willing to answer any questions, no matter how trivial or foolish, I would have wandered lost in the world of Hilbert space. To Carmichael, who helped highlight and spark my interest about the paradoxical world of quantum reality. Without his insight into what actually went on behind the mathematics, I would have never had the pleasure of discovering the interesting phenomena detailed within.

On a wider spectrum, this thesis could not have existed if I were sucked away into the evil world of Engineering. The physics department has been exceptionally supportive since I walked apprehensively into the hallways as a high school student, seeking advice on what course I should take. Out stepped Gary Bold, radiating enthusiasm from every fibre of his being. The tour of the atom trapping labs and his reassurances that I could immediately skip Stage I Physics was to say the very least, inspiring. Even now, I can picture him, kneeling on the ground chanting, $\mu_0 I$, over 4π.

A Master's in the eccentric world of Physics would not have been complete, without a world of equally eccentric people. In the process of writing my thesis, a lot of interesting ideas sparked from conversations with my office mates. To Sam, Tara and Matthew in the graduate maths room, who not only provided interesting diversions, but also, sometimes, even helpful advice! My knowledge of group theory and functional analysis was never exceptional, and a perspective of one who can related to the formulae is invaluable. In this, I should also thank Austin, the lecturer whose advice of keeping a pet mathematician could not be more correct.

To Andy and Chang in the 'teleportation chamber', who not only provided interesting conversation, but also helped me battle against the great evils of computer glitches, demented cyberbugs and Latex gone haywire. And to Finn and Robert, who were always there to offer laughing sympathy on Thursday afternoons, when I frantically prepared the material for my weekly progress report.

Numerous friends have offered their help and support throughout my research. To Erin Higgins, whose numerical skills helped immensely when wading through the swamps of linear algebra. To Lydia Philpott, whose commendable habit of placing 25 books in quantum superposition and assimilating them simultaneously surely helped in suggesting some of the more appropriate quotes featured within this thesis. To Amy Peng, for her clear and coherent notes that were easy to both understand and follow.

In the process of writing the thesis, one cannot not stress how useful it is to have interested listeners. Only when one tries to explain a concept, does one realize what is interesting, what is exciting, what is confusing, and what equations cause people to flee in abject terror. My thanks to Christian Weedbrook, for his questions on entanglement and for his ability to apply the theories of quantum optics into a game pool or a dish of watermelon. To Sarah Midgeley, the only person I know to refer to a theorem as being 'cute', whose child-like wonder motivated me to conjure many of the more creative examples featured within this thesis. To everyone who visited and listened to my poster presentation at the Physics A.I.P. Conference, and my MSc. presentation here in the University of Auckland, the very experience has helped me organise my chaotic thoughts.

When the final draft of this thesis was finally complete, I recall saying apologetically to my supervisor, "There's probably plenty of mistakes within this thesis, I read through it last night and could find around one per pager." To which he heartily replied to the effect of, "There's a lot more than that!". So, to my supervisors, Parkins and Carmichael for taking so much time to help me nail out all the epic typos planted within this thesis. Also to Mathew and Tara, graduate students who sacrificed a stay of assignments to help proofread my thesis. To dad, who took the effort to read half of this thesis, despite not understanding any of the mathematics. I am indebted, for without them, you could be reading about 'Harmonic Isolators', 'Faraday Oscillators', or that 'Quantum Mechanics' is fundamentally a marketplace, that is 'bazaar'.

Last, but not least, I'd like to thank my family, and parents, for supporting me throughout these academic years, sparing me from much of the daily chores so that I could have luxury acting as the absent minded professor.

Mile Gu
March 29, 2005

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Chapter 1

Introduction

> "The imagination, lost in the immensity of the universe, will have difficulty to conceive its bounds."
> - *System of the World, P. Laplace*

1.1 A Paradox

In a galaxy far far away, Alice and Bob are on separate spaceships, travelling away from each other infinitesimally close to the speed of light. Prior to their departure, they shared the dissociation of some diatomic molecule. By the conservation of angular momentum, the spins of the dissociated atoms must always be equal and opposite.

If we assume the atoms to be spin-$\frac{1}{2}$ particles, then a measure of spin, regardless of direction, for each individual atom will have only two outcomes $|\uparrow\rangle$, and $|\downarrow\rangle$. In Dirac notation, the state of the system is just a superposition of $|\uparrow\rangle|\downarrow\rangle$ and $|\downarrow\rangle|\uparrow\rangle$

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B\right) \tag{1.1}$$

One night, Alice decides to measure the spin of her particle in a direction we shall call z. If she were to get the result $|\uparrow\rangle$, she would automatically deduce that the state of Bob's particle would be $|\downarrow\rangle$, and indeed, any measurement performed by Bob afterwards would certainly conform to Alice's expectations. But, wait a minute! How do we define 'afterward', when Alice and Bob reside in causally disconnected regions of space-time?[1]

This question should come as no surprise to the reader, for we deal with conditional probabilities in our daily lives. If you know that there exists only one George Bush in the universe and spotted him in New York, you can deduce that Bush would not be in Auckland at the same time[2]. In fact, you can even deduce that any experimenter in Auckland who performs

[1] If we wish to be pedantic, Alice and Bob's ships are not entirely disconnected as they can never reach the speed of light with respect to each other. However, we can consider them travelling fast enough that for all intents and purposes, this can be considered true.

[2] By 'At the same time', we mean of course, two events in space like domains with respect to each other.

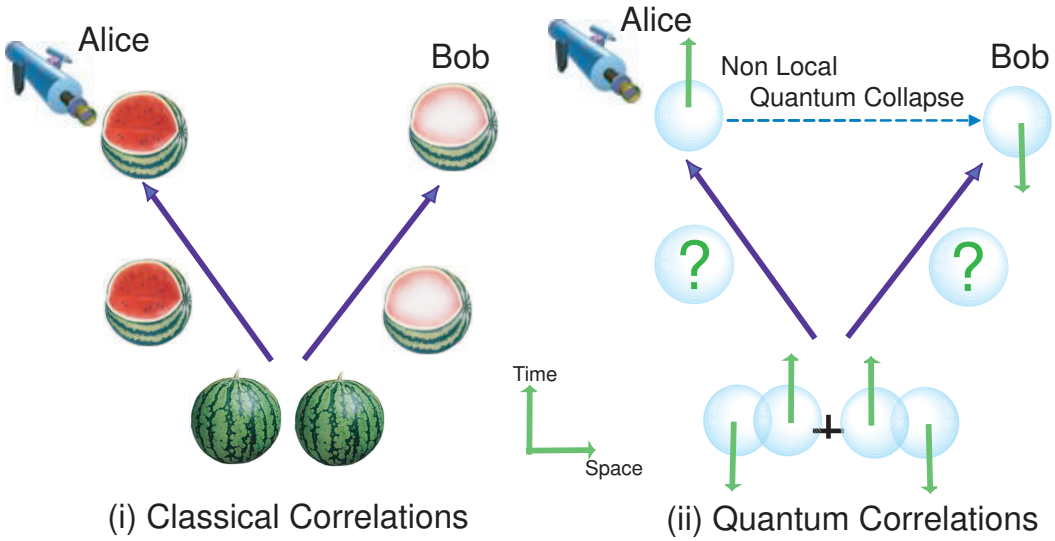

Figure 1.1: The space time diagrams for objects, classic variables(i) and quantum variables(ii) shared between Alice and Bob. In the classical case of watermelons, the states of each watermelon is determined prior to separation, and thus Bob's measurement result is not affected by Alice's measurement. In the case of spin half particles, however, neither particle has a well defined spin till measurement. Hence Alice's measurement does affect Bob, despite the fact he is in a causally disconnected space-time region.

a 'detect Bush' experiment would return a null result, even if the experiment was performed at a point in space-time not causally connected to you.

Similarly, suppose Alice and Bob were to have shared two watermelons prior to their departure, only one of which is ripe. If Alice were to cut her watermelon open, and find it to be ripe, she would have every reason to deduce that Bob took the non-ripe watermelon. And indeed, if Bob were to perform the measurement on his melon, he would find it to be non-ripe, despite the fact that he and Alice live in reference frames that are causally independent. There is however, no causality paradox, for the states of watermelons were predetermined ever since their separation. Bob's measurement result did not depend upon Alice's measurement, but rather his choice of watermelon prior to departure, an event that is in the past region of space-time for both Alice and Bob (Figure 1.1 (i)).

In the formalism of quantum mechanics, however, states are not necessarily predetermined. In particular, they are only defined when they are measured. A spin-half particle with a definite spin in the z direction will be in an equal quantum superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$ in both the x and y directions.

Alice did not need to restrict her measurement to the z direction. She could have performed

the same experiment with respect to either direction x, or y, giving her exact knowledge of the spin of Bob's particle in either of these directions. This suggests that prior to Alice's measurement, there were already well defined values of spin for Bob's particle in any of the three directions. Yet, in the framework of quantum mechanics, states with well defined non-zero spins in all three directions do not exist.

One is led to deduce that the act of the measurement by Alice has caused 'instantaneous' collapse of the wave function that describes the state of Bob's particle, in which case the values of spin that Alice has deduced are not true values of Bob's original state. But, instantaneous? Such a word cannot be defined in the framework of relativity. In particular, Alice and Bob are causally disconnected and we can always choose a frame of reference such that the collapse of Bob's state occurs before the measurement made by Alice. The embarrassing dilemma of cause after effect is suddenly upon us (Figure 1.1 (ii)).

It was 1935 when Einstein, unsatisfied with the probabilistic and non-deterministic nature of quantum mechanics, published a variation of the above argument with Podolsky and Rosen [1]. Their aim was to use the conclusion of non-causality to shown that quantum mechanics could not be a complete physical description of reality, that the spin-half particle were like the watermelons of classical physics, with well defined parameters prior to measurement.

It was not until 1964 that Bell proposed a set of inequalities show that Einstein's hope for a completion of quantum mechanics that respected local realism could not be fullfilled[2]. The string of ever more precise experiments that followed has verified that quantum mechanics was indeed complete [3], and it was the notion of local reality that we need to abandon. The quantum world does permit 'eerie action at a distance'.

That is, quantum collapses can be caused by measurements that occur in space-like domains of space-time. A state function can collapse in a reference frame prior to the measurement that caused its collapse. The universe is either non-local or non-causal, and when Alice performs that measurement of her particle, her measurement result will affect Bob's, regardless of the space-time relation.[3]

Quantum entanglement, coined to describe this phenomenon of non-local relations, is an entity that remains shrouded in mystery. We know that entanglement, being a property independent of its physical manifestation, shares similarities with other physical resources, such as energy, and yet also exhibits strange properties unique to quantum systems. In particular, we still have little idea how to characterize entanglement between more than two objects, and even entanglement of just two objects is still not well understood when classical probability is also involved.

Yet, entanglement has become much more than a physical curiosity. Current advances in quantum technology, from quantum computing[9] to quantum teleportation[4], all rely on the either the non-local or non-realistic nature of the universe. In order to fully understand these properties, we will need a complete theory of entanglement, much as one could not build modern electronics without a complete understanding of electricity.

[3]It is a misnomer however, that 'affect' is equivalent to information transfer. As wave functions cannot be directly detected, affecting a wavefunction does not equate to transfer of information.

While a complete analysis and formulation of quantum entanglement is deserving of a Nobel Prize, this thesis is much more modest. It has two primary aims. The first is one of pure physics, to achieve a greater understanding of how the universe works; the second is more applied, to use this knowledge to construct systems of practical use.

In the first part, we will consider how one can characterize the entanglement of quantum states that also feature classical uncertainty. We will see that this is the key to understanding how entanglement evolves in open systems, where interactions with the environment make additional classical uncertainty unavoidable. Such analysis will demonstrate many of the unique and peculiar wonders of entanglement, and its related paradox, the measurement problem. In particular, we will observe how the evolution of a system can be affected significantly by measurements performed on what it has already emitted, much as how the atomic state of Bob's particle could be altered, despite the fact that the event that caused the alteration is causally disconnected.

In the second part, we will apply these results to the area of entanglement engineering. In particular, we will show how variations of a particular open system can be used to generate a wide range of different entangled states, which can then be used for quantum protocols, such as teleportation and quantum information processing.

1.2 Outline of Thesis

This thesis is essentially separated into two partitions. The first looks at the theoretical framework on which we can understand the nature of entanglement within open systems, while the second considers applying this knowledge to generate entanglement for quantum protocols.

As the contents of this thesis are based on a wide range of background knowledge, it was difficult to decide what to include. In our final considerations, we have decided to focus our background on areas such as information theory with which the reader is less likely to be familiar. In other topics, we will assume that the reader already has a passing understanding, and will only need to be briefly refreshed.

For any quantity to be subjected to mathematical analysis, there needs to be a way to quantify it. Chapter 2 begins our journey to quantify entanglement in a physically meaningful manner. We outline the well known Von Neumann measure for analyzing entanglement of pure states and show how it can be introduced from a physical perspective.

Chapter 3 introduces the theory for open systems, and how classical uncertainty in such systems cannot be avoided. This leads to the idea of mixed states, and the ambiguities which arise when we try to quantify the entanglement of such states. We will then present a promising solution to the problem using quantum trajectories.

The approach of quantum trajectories is best demonstrated explicitly, and we do so by considering how it helps unravel the Cascaded Cavity System.[27]. Chapter 4 outlines this system.

Chapter 5 is a culmination of the previous chapters, as we apply methods introduced in Chapter 3 to analyze how entanglement evolves within the cascaded cavity system. We will show that under appropriate conditions, the system can exhibit stochastic cycles between fully entangle Bell states, a phenomenon that cannot be observed using conventional methods.

Chapter 6 shifts the focus to our second objective, to use the techniques developed to generate entanglement between quantum systems of arbitrary dimensions. Mathematical methods are used to completely characterize the steady state of a general cascaded atom-cavity system, where we show that the entanglement is caused by a parity displacement that does not scale well with the dimensionality of the system.

Chapter 7 continues our search for a viable way to generate entanglement between higher dimensional systems. We present a bosonic analogy to the cascaded cavity system, and show how this analogy leads us to an effective solution.

Chapter 8 summarizes the work that we have presented, and details possibilities for future research in this area that can be made. In particular, we consider the nature of entanglement within open systems, and whether the lack of local reality forbids us to completely characterize the entanglement within an open system.

1.3 Pathways through this thesis

Since this thesis is a combination of two distinct objects, there are actually two different and reasonable independent pathways through this thesis. The first details the fundamental nature of entanglement within open systems, while the second considers practical proposals in the field of entanglement engineering.

Those interested primarily in fundamental aspects of entanglement should read Chapter 2 and 3, but the technical derivations of the Cascaded Cavity System in 4 is probably unnecessary. Chapter 5 is the essential focus of these ideas, and helps elucidate, by example, what quantum trajectories can tell us about entanglement within open systems. Finally, Section 8.2 details the implications of the study, with respect to whether we can attribute exact physical values of entanglement to such systems.

Alternatively, the bulk of the material on Entanglement engineering is detailed in the two separate proposals made in Chapters 6 and 7. Readers primarily interested in this area could omit much of the detail given in Chapter 3. The conclusions to our progress made within this field are made in Section 8.2.

Chapter 2

Quantum Entanglement

> "I would not call that *one* but rather *the* characteristic trait of Quantum Mechanics, the one that enforces departure its entire departure from classical thought."
> -*E. Schrödinger about Entanglement*

2.1 Introduction

Physics is a quantitative science, a science that predicts experimental results in the form of numbers. That is, every quantity in Physics can be quantified, so that theory is compared to experiment via standard errors. In order to describe the evolution of some physical quantity, physics uses the language of mathematics.

Throughout history, physics has managed to quantify each physical property of interest, starting from the very simplest notions of distance, to more esoteric quantities such as temperature. For each quantity, we need not only develop the idea of a totally ordered set, so that we can say X is longer Y, but also a fundamental unit of measure, so that we can say X is twice as long as Y.

A weather broadcaster on 3 News once stated,"Tomorrow for Dunedin will be twice as hot, doubling from 5 to 10 degrees..." which made no physical sense[1]. Indeed, prior to the development of the Kelvin scale, there was no fundamental measure, and it was physically incorrect to say that anything was twice as hot as anything else.

One of the most elusive quantities of interest in modern physics is quantum entanglement, whereby, we say that two objects are entangled if the state of the composite system cannot be factored in terms of states for the two individual objects. While this definition provides a qualitative definition, it does not tell us anything about how much entanglement exists. Could it, for example, make sense to say $|\phi\rangle$ is twice as entangled as $|\psi\rangle$?

In this chapter, we describe how a fundamental measure of quantum entanglement can be

[1] At least, for the poor people living in Dunedin, I sincerely hope it was not true.

formulated, and at the same time gain a grasp of what entanglement means physically. We will see that this abstract quantity is closely tied with notions of uncertainty and information loss. Therefore, we will take a brief survey of classical information theory, and extend the notions of classical information into the quantum world.

2.2 Shannon Entropy

Before we introduce the idea of entropy for quantum information, we will first analyze an analogous case in classical theory.

Consider a fair coin and a fair 4-sided die. Suppose someone were to toss the coin, or roll the die, and record the results on a sheet of paper. One could ask the question, which of these two recordings contain the most information, or more precisely, how much information do you gain from knowing nothing about the result, to reading the results on that piece of paper? How many times more or less information does one gain from results of the die roll compare to that of a coin toss? Alternatively, what is the uncertainty introduced by your lack of knowledge of these results?

Without a fundamental measure for information, such questions cannot be answered, or even defined. Thus, the first step is to develop an absolute measure of this quantity. In this case, the bit is an obvious candidate.

Consider the bit; it tells us whether one binary variable is 0 or 1 given no prior knowledge of that variable. Reversing the argument, we can clearly see that the knowledge gained from learning the result of a toss of a fair coin is exactly one bit.

One can simulate a 4-sided dice by tossing a coin exactly twice;[2] as such we can say that it takes exactly two binary decisions, or two bits of information, to learn the result of measuring a uniformly distributed random variable of four states.

Now consider a random variable X with four states, such that $P(X = a) = \frac{1}{2}$, $P(X = b) = \frac{1}{4}$, $P(X = c) = P(X = d) = \frac{1}{8}$. The measurement of X can be represented via a sequence of binary decisions demonstrated in Fig. 2.1. We can compute that the

$$\text{Expected Number of Decisions} = 0.5 \times 1 + 0.25 \times 2 + 0.25 \times 3 = 1.75. \tag{2.1}$$

The information gained upon knowledge of X is exactly 1.75 bits. This is less than 2 as the random variable has a non-uniform distribution, signifying prior knowledge of how the variable could behave. Due to this prior knowledge, the measurement of the variable will give less information than if we had no information whatsoever to begin with. Alternatively, we can view this in terms of uncertainty. X contains less uncertainty than a uniform distribution, as it is intrinsically biased towards certain states.

This allows us to give a formal definition of Shannon Entropy [6].

[2]For example, the result of the first toss could indicate whether the number was even or odd, and the second determine which even (odd) number it is.

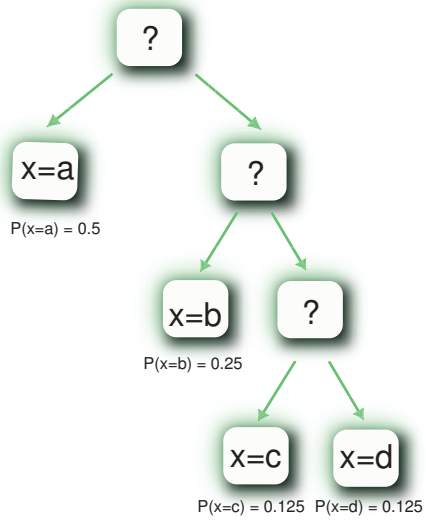

Figure 2.1: Binary decision tree for a random variable X

Definition 1 (Shannon Entropy) *Consider a random variable X that can assume states $i = 1, \ldots, N$ with probabilities p_i. The Shannon Entropy, defined as the amount of information, in bits, gained upon knowledge of X is given by*[3]

$$S(X) = -\sum_{i=1}^{N} p_i \lg p_i. \tag{2.2}$$

Alteratively, $S(X)$ represents the degree of uncertainty we have about X.

Suppose a person attempts to encode an alphabet such that each letter has a random distribution X in terms of binary digits, then one can visualize $S(X)$ as the average number of bits required per letter to encode a piece of writing written in this alphabet.

Shannon Entropy provides a fundamental way of measuring information, and using this, it makes sense to say that knowledge of the result of tossing a fair 2^N sided dice contains exactly N times as the information of the result of a coin toss. This measure has become extremely important in classical communication theory, and statistical mechanics, as it provides the first quantitative, and physical measure for a previously qualitative property. In the former, it is a fundamental tool for analyzing communication protocols [7], and in the latter, it gives a measure of Gibbs Entropy, in units of Boltzmann's Constant k_B [8].

[3]We use the standard notation of theoretical computer science. Where lg denotes $\log_2$.

2.3 Von Neumann Entropy

The ideas of classical information theory can be extended into the quantum world, where we ask how many classical binary decisions it takes to discern the quantum state of a system. Whereas the Shannon Entropy measures the classical uncertainty associated with a classical probability distribution, the Von Neumann Entropy measures the classical uncertainty in an ensemble of quantum states.

Consider, for example, a quantum ensemble that is an equal mixture of $|0\rangle$ and $|1\rangle$. As the two states are orthogonal, we can consider them exactly as if they were just classical states, and we find that the Von Neumann Entropy is exactly 1.

In the more general case, a quantum ensemble composed of a mixture of pure states $|\phi_i\rangle$ with probabilities p_i is represented by a density matrix

$$\rho = \sum_{i=1}^{N} p_i |\phi_i\rangle\langle\phi_i|. \tag{2.3}$$

This leads us to a general definition [9].

Definition 2 (Von Neumann Entropy) *Consider a quantum ensemble described by a density operator ρ. We define the Von Neumann Entropy as*

$$S(\rho) = -\mathrm{Tr}(\rho \lg \rho) = -\sum_i \lambda_i \lg \lambda_i, \tag{2.4}$$

where λ_i are the eigenvalues of ρ.

We see that this definition satisfies the classical correspondence principle for the case where all the states $|\phi_i\rangle$ that make up the ensemble are orthogonal, whereby ρ is diagonal in the $|\phi_i\rangle$ basis with eigenvalues $\lambda_i = p_i$; thus,

$$S(\rho) = -\sum_i p_i \lg p_i = E(p). \tag{2.5}$$

The quantum nature comes into play when states within the ensemble are not orthogonal, and hence cannot be treated as classical states.

2.4 Introduction to Quantum Entanglement

From undergraduate physics, we know that one of the major features of Quantum Mechanics that has no classical analogy is Entanglement. This section will outline this phenomenon, and how one can put the phenomenon in a quantitative framework.

2.4.1 What is Entanglement?

Unlike length or temperature, quantum entanglement is not something that be can physically perceived with our senses. Thus, the first step in quantifying a phenomenon is to be able to define it, and have an intuition on what it is.

Recall that in our introductory example, the states of Alice and Bob were exhibited entanglement, since a measurement performed on one of the two particles causes a non-local collapse in the state of the other. In a sense, each particle contained no independent reality. That is, despite the fact we had complete characterisation of two particle the system, we could not describe the states of either particle separately.

This concept is entirely non-trivial. Suppose there were two watermelons, each of which can be either ripe or not ripe. Regardless of what state the system is in, the system can always be described by first stating the state of the first melon, followed by that of the second. If the same procedure was attempted on a Bell state, we're instantly met with trouble. We have already observed that each particle exhibited no local reality, and had no definite internal state till the point of measurement. That is, the reality of the two particles are intertwined, and cannot be factored into local realities corresponding to each individual particle.

Definition 3 (Entanglement) *Consider the state $|\phi\rangle$ of a composite system composed of subsystems A and B. We say that A and B are entangled if there do not exist states $|\psi\rangle$ and $|\varphi\rangle$ of the two individual systems that factor $|\phi\rangle$. That is:*

$$|\phi\rangle \neq |\psi\rangle \otimes |\varphi\rangle \qquad \forall |\psi\rangle, |\varphi\rangle \tag{2.6}$$

Conversely, the system is separable, if $|\phi\rangle$ can be written as a tensor product of the two subsystems. That is $|\phi\rangle = |\psi\rangle \otimes |\varphi\rangle$ for some $|\psi\rangle$, $|\varphi\rangle$.

2.4.2 A Quantitative Definition

In thermodynamics, we wish to compare the temperature of two objects, in kinematics, we wish to compare their velocity, for doing so allows us to make quantitative predictions. Quantum information is no different, with teleportation protocols etc. all requiring sources of entanglement to function. Yet, how 'strong' does that source need to be? How much of that source is required? These questions cannot be answered, or even made sense of, until we can quantify entanglement.

Consider Alice and Bob, who share the two qubits of the Bell State

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.7}$$

Now suppose Bob were to make a projective measurement on his qubit with respect to the basis $|0\rangle$ and $|1\rangle$; he would project the system into an equally mixed ensemble of $|00\rangle$ and $|11\rangle$, corresponding to a measurement result of 0 and 1 respectively.

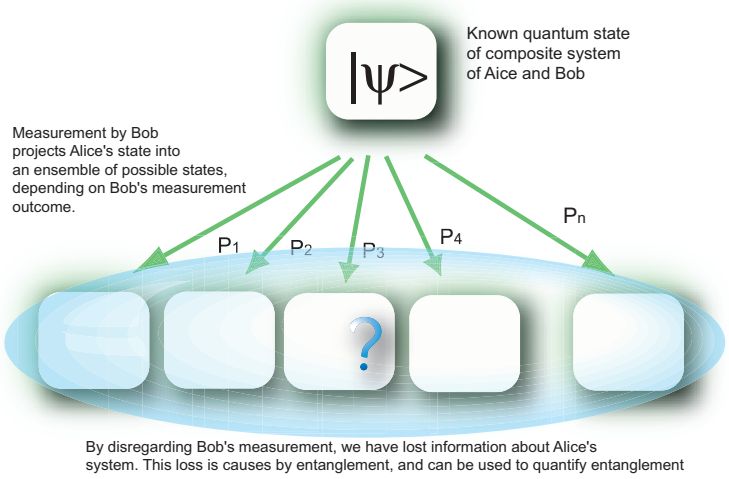

Figure 2.2: The connection between entanglement and information loss.

As Alice is unaware of Bob's measurement result, her qubit would be an equal mixture of $|0\rangle$ and $|1\rangle$. If we consider this physically, it becomes apparent that the entangled nature of Alice and Bob's system has caused Alice to lose information about her own system when she disregards the actions of Bob. That is, the reality of the two systems are somehow intertwined (See Figure 2.2)

From this perspective, the amount of information loss due to neglecting Bob's system would make perfect sense as the measure of entanglement.

Definition 4 (Entanglement) *Given a pure state $|\phi\rangle$ of a composite system consisting of subsystems A and B, entanglement is defined as the amount of* information loss *in the Von Neumann sense, about the true state of A if one disregards projective measurements on B.*

Of course, this definition only makes sense if it is independent of which projective measurement Bob performs on his system. We shall prove this in the next section.

2.4.3 Calculating Entanglement

It is all well and fine to give a formal definition of entanglement. However, how does one go about calculating it? In this section, we will derive an explicit formula consistent with the definition given above.

Consider a pure state of the composite system $A \otimes B$ expanded in the form

$$|\phi\rangle = \sum_{i,j} c_{i,j} |\phi_i\rangle_A |\phi_j\rangle_B, \tag{2.8}$$

where $\{|\phi_i\rangle_A\}$ and $\{|\phi_j\rangle_B\}$ are sets of orthogonal states in subsystems A and B respectively.

Now suppose Bob makes an arbitrary projective measurement on his subsystem. We can assume this measurement was made with respect to the basis $\{|\phi_j\rangle_B\}$ without loss of generality[4]. Such an action projects A into an ensemble of states, such that

$$|\phi\rangle_A = |\psi_j\rangle_A = \mathcal{N}\sum_i c_{i,j}|\phi_i\rangle_A$$

with probability $p_j = \sum_i c_{i,j}$

We can represent this ensemble via the density operator

$$\rho_A = \sum_l p_l|\psi_l\rangle_A\langle\psi_l|_A \tag{2.9}$$

$$= \sum_{i,k}\sum_l c_{i,l}c^*_{k,l}|\phi_i\rangle_A\langle\phi_k|_A. \tag{2.10}$$

While this equation appears complex, it has a simple interpretation. If we let $\chi = |\phi\rangle\langle\phi|$ be the density operator of the composite system, then

$$\begin{aligned}\mathrm{Tr}_B\chi &= \mathrm{Tr}_B\left[\sum_{i,j}\sum_{k,l} c_{i,j}c^*_{k,l}|\phi_i\rangle_A|\phi_j\rangle_B\langle\phi_k|_A\langle\phi_l|_B\right]\\ &= \sum_{i,j}\sum_{k,l} c_{i,j}c^*_{k,l}|\phi_i\rangle_A\langle\phi_k|_A\left(\langle\phi_l|_B|\phi_j\rangle_B\right)\\ &= \sum_{i,k}\sum_l c_{i,l}c^*_{k,l}|\phi_i\rangle_A\langle\phi_k|_A\\ &= \rho_A\end{aligned} \tag{2.11}$$

That is, regardless of which projective measurement Bob decides to make, the resulting mixed ensemble on Alice's side would always form the same density operator. Furthermore, this density operator is exactly the partial trace of the original system.

Proposition 1 (Projective Measurements and the Partial Trace) *Suppose $|\phi\rangle$ is the state of the composite system $A\otimes B$. Then any projective measurement on B will project the subsystem A into an ensemble with density operator $\rho_A = Tr_B\left(|\phi\rangle\langle\phi|\right)$.*

Now as the Von Neumann Entropy is defined by the density operator of the partial trace, it is clearly independent of Bob's choice of measurement. Thus, we can truly interpret entanglement as the amount of information loss to one subsystem, after disregarding the other.

This argument also leads us to a simple equation for Von Neumann Entanglement [20].

[4]This is true since our choice of basis representation for $|\phi\rangle$ is also arbitrary.

Theorem 1 (Von Neumann Entanglement) *Given a pure state $|\phi\rangle$ of a composite system consisting of subsystems A and B, the Von Neumann Entanglement can be computed via the formula*

$$E(|\phi\rangle) = Tr(\rho_A \lg \rho_A), \tag{2.12}$$

where $\rho_A = Tr_A(|\phi\rangle\langle\phi|)$

Applying this theorem to our example above, we find that the Entanglement of this Bell state is exactly 1. It is a simple exercise to demonstrate that this calculation holds true for all Bell States.

Proposition 2 (Bell State Entanglement) *The Von Neumann Entanglement of a Bell state is exactly unity.*

2.4.4 Conservation Properties of Entanglement

For our measure of entanglement to have some physical significance, it needs to have well defined physical properties. For example, our measure of distance makes sense as placing two objects of length l_1 and l_2 together gives something of length $l_1 + l_2$.

In this section, we demonstrate that the Von Neumann measure indeed makes sense for pure bipartite entanglement via a concise outline of the detailed treatment by Nielson and Chuang [9].

Consider again the measurement of length and its standard unit of measure in terms of the meter. What does it mean for a plank of wood to be 7 meters long? Well, a simple definition would be that the same plank can be broken up into 7 pieces, each a meter in length.

Things get slightly trickier when the plank is of non-integer length. Suppose the plank is x meters long, and $x = p/q$ is rational. Then a more general condition is that q planks of the same length can be converted into $p = xq$ meter size pieces. In the final generalization when q is an arbitrary real number, we use:

Definition 5 (The Measure of Length) *We say an object is x meters long, where x is real, if n such objects can be divided into nx meter sized objects in the limit that $n \to \infty$.*

Sounds simple, doesn't it?

It turns out that there is a very similar theorem for entanglement.

Theorem 2 (Entanglement Interchange) *Consider an arbitrary pure state of a composite system $|\phi\rangle$. It is possible to convert n such copies of $|\phi\rangle$ into $nE(|\phi\rangle)$ Bell states, via the use of local operations and classical communications, in the limit that $n \to \infty$. Similarly, one can convert $nE(|\phi\rangle)$ Bell states into n copies of $|\phi\rangle$ for any $|\phi\rangle$.*

A proof of this theorem is rather involved, and details can be found in [9].

Comparing this to our definition of a measure of distance, we can easily draw analogies to see that the Von Neumann measure for entanglement does indeed make sense for pure states. In addition, the Bell State makes a natural fundamental unit of entanglement, and the Von Neumann measure determines exactly the number of Bell States worth of entanglement a given state has.

2.5 Entanglement of Special States

In this section, we will apply our measure to evaluate the entanglement of certain special states. The goal here is two-fold, to provide a nice demonstration of our measure, and results that will be used later in this thesis.

2.5.1 Optimal Entanglement of Qudits

A qudit is a quantum variable with d states, $|1\rangle, |2\rangle, \ldots, |d\rangle$. Let us consider two qudits in the entangled state of form

$$|\phi\rangle = \sum_{i=1}^{d} |i\rangle|i\rangle. \tag{2.13}$$

An unknown measurement on Alice's qudit in the number state basis will project Bob's qudit onto an even mixture of $|1\rangle, |2\rangle, \ldots, |n\rangle$. Applying (2.5), we find

$$E(\phi) = \sum_{i=1}^{d} \frac{1}{d} \lg d = \lg d \tag{2.14}$$

Since any quantum variable of d dimensions has at most entropy $\lg d$, $|\phi\rangle$ is a maximally entangled state of two entangled qudits[5]. That is

Theorem 3 *The maximum entanglement shared by two qudits with respect to the Von Neumann measure is exactly* $\lg n$

This simple result allows us to define how efficiently two qudits are entangled with each other.

$$\text{Entanglement Efficiency} = \frac{E(|\phi\rangle)}{\lg d} \tag{2.15}$$

This measure will become very useful later, when we consider general protocols for creating entanglement between qudits.

[5] A rigorous proof of this can be made using the concept of relative entropy [9].

2.5.2 Entanglement of a Two Mode Squeezed State

In Quantum Optics, a very useful measure of entanglement between two field modes is that of two-mode squeezing. Here, a state is consider squeezed if the X and Y quadratures of each mode,[6] which we shall denote by X_1, X_2, etc. satisfy the properties

$$(\langle X_1 \pm X_2 \rangle)^2 = e^{\pm 2r} \qquad (\langle Y_1 \pm Y_2 \rangle)^2 = e^{\mp 2r}. \tag{2.16}$$

where r is the squeezing parameter. In the number state basis, the two mode squeezed state is expressed as

$$|\phi(\lambda)\rangle = \sqrt{1-\lambda^2} \sum_{n=0}^{\infty} \lambda^n |n\rangle_A |n\rangle_B. \tag{2.17}$$

where $\lambda = \tanh r$ [10]. Suppose there is no squeezing, then $\lambda = 0$, and we get the vacuum state for two modes. Suppose there is infinite squeezing, whereby $\lambda = 1$, then we get an equal superposition of all correlated states $|n\rangle|n\rangle$ that resembles the maximally entangled state of two qudits, where $d \to \infty$.

Suppose Alice made a projective measurement to her mode with respect to the number basis, she would project Bob's state into a weighted mixture of all number states. As the number states are orthogonal, (2.5) can be immediately applied to give

$$\begin{aligned}
E(|\phi\rangle) &= -(1-\lambda^2) \sum_{n=0}^{\infty}, \lambda^{2n} \lg \left[\lambda^{2n}(1-\lambda^2)\right] \\
&= -(1-\lambda^2) \left[\lg(\lambda^2) \sum_{n=0}^{\infty} n\lambda^{2n} + \lg(1-\lambda^2) \sum_{n=0}^{\infty} \lambda^{2n} \right], \\
&= -\frac{\lambda^2}{1-\lambda^2} \lg(\lambda^2) - \lg(1-\lambda^2) \qquad (2.18) \\
&= -\sinh^2(r) \left[\lg(\sinh^2(r)) - \lg(\cosh^2(r))\right] + \lg(\cosh^2(r)) \\
&= \cosh^2(r) \lg(\cosh^2(r)) - \sinh^2(r) \lg(\sinh^2(r)) \qquad (2.19)
\end{aligned}$$

This is in agreement with the standard result [11].

It is worth noting that in the case of infinite squeezing, $r \to \infty$, the entanglement tends to infinity. This is logical, given that each mode contains an infinite number of states, and hence optimal quantum correlations allow the extraction of an infinite number of Bell s'tates.

[6] A concise exposition of continuous variable quantum states can be found in Appendix B.

Chapter 3

Entanglement in Open Systems

> "Not to be absolutely certain is, I think, one of the essential things in rationality."
> -*Bertrand Russell* [34]

3.1 Introduction

Physics is a process of mathematical abstraction. Given a system, physics seeks to find the underlying mathematical principles, then use them to build an analytical model which converges to that of the actual system of interest under reasonable approximations.

One major approximation, made in the analysis of classical systems, is the abstraction of their interaction with the environment. That is, we take the system of interest, and make vast simplifications to how it affects the environment, and how the environment affects it, with separate realities.

When a watermelon flies through the air, we simplify its collisions with the atmospheric particles by the introduction of a constant of friction. When the same watermelon is placed into a heat bath, we assume that the heat bath is large enough so as to not undergo a change in temperature itself as the melon settles to thermal equilibrium. At all times, we assume that the watermelon and its environment are separate entities, each of which can be described individually.

These assumptions are justified, and very necessary. For the only closed system in the universe, is the universe; and while an analysis of the entire universe would certainly be ambitious for a Masters Thesis, it is better left as an exercise for the supremely talented reader.

Yet, quantum mechanics deals with interactions at a fundamental level, and the very fact that it is a microscopic theory makes many typical macroscopic approximations invalid. While it is often still true that our system makes little impact on the environment, the very act of a measurement by the environment can cause the collapse of a wavefunction in our system.

What if that watermelon had an intrinsic spin of $\frac{1}{2}$, and upon collision with another spin half particle, became entangled with it in a Bell state? Would not a measurement on the other particle then have serious effects on the state of the watermelon? The entanglement of system and environment has caused loss of information of the system, a loss of information that can affect the evolution of entanglement within the system itself.

As we wish to track the evolution of entanglement, it is essential such quantum information is retained. Somehow, we will need to take into account interactions with the outside world, without explicitly including the outside world in our model. This chapter will outline these complications and a technique used to address them.

3.2 The Quantum Theory of Open Systems

An open system is a system that interacts with its environment. For practical purposes, almost any physical system of interest is open. The swinging pendulum is colliding with air particles, and the tip of its string grinding against its axel. Even in perfect vacuum, with complete absence of friction, the fact that you can observe the pendulum implies that it is under a constant, unrelenting barrage of photons.

While classical physics allows us to ignore these interactions, in a quantum system involving photons, such events can obviously not be ignored! That is, for any system to be meaningful to the outside world, we need to interact with it in some manner, and thus any observed system is open.

In this section, we will briefly first outline the stand approach to address such systems, in a method presented by Carmichael [12].

3.2.1 The Exact Formulation

In order to have an accurate quantum description of an open system, we will not be able to disregard the environment in such a cavalier fashion. Hence, let us first consider a fully quantized model, where the Hamiltonian has the general form

$$H = H_S + H_E + H_{SE}. \tag{3.1}$$

H_S is the Hamiltonian of our system of interest, H_E that of the environment, and H_{SE} the coupling between the two. Let us also define $\chi(t)$ as the density operator for the system plus environment, which satisfies Schrödinger's equation:

$$\dot{\chi}(t) = \frac{1}{i\hbar}[H, \chi] \tag{3.2}$$

We can rewrite this equation in the interaction picture by absorbing the local evolution to

$$\dot{\tilde{\chi}}(t) = e^{(i/\hbar)(H_S+H_R)t}\chi(t)e^{-(i/\hbar)(H_S+H_R)t}, \tag{3.3}$$

whereby the evolution for $\tilde{\chi}(t)$ is governed by

$$\dot{\tilde{\chi}}(t) = \frac{1}{i\hbar}[H_{SE}, \tilde{\chi}]. \tag{3.4}$$

We can write a formal solution for this equation:

$$\tilde{\chi} = \chi(0) + \frac{1}{i\hbar}\int_0^t dt' \left[H_{SE}(t'), \tilde{\chi}(t')\right]. \tag{3.5}$$

Finally substituting the above into (3.4) allows us to rewrite the dynamic equation in a form that is much more suitable for planned approximations [12].

$$\tilde{\chi} = \chi(0) + \frac{1}{i\hbar}\left[H_{SE}(t), \chi(0)\right] - \frac{1}{\hbar^2}\int_0^t dt' \left[H_{SE}(t), [H_{SE}(t'), \tilde{\chi}(t)]\right], \tag{3.6}$$

This equation is exact, and will allow us to isolate our system of interest in a systematic manner.

3.2.2 Removing the Environment

The environment is extremely complex, so complex that it requires a quantum computer the size of the universe to simulate. Without a spare customizable universe at our disposal, the best we can aim for is to remove the environment from our state description, but still include its effects on our system of interest as best as possible.

Suppose that the system and its environment were isolated from each other prior to the start of the experiment. That is, they were set up separately. Then, it is logical to assume that at $t = 0$, the state of the environment and system is not entangled. Explicitly, we have a product state of the form

$$\chi(0) = \rho(0)E(0), \tag{3.7}$$

where $E(0)$ is the initial density operator of the environment, and $\rho(0)$ that of the system.

In order to describe the system alone, we wish to remove the environment from our formulation at a general time t. The logical way to do this is to consider only the information we know about the system regardless of whatever occurs in the environment. From *Proposition 1*, an accurate mathematical description of such an object is the partial trace

$$\rho(t) \equiv \mathrm{Tr}_E[\chi(t)]. \tag{3.8}$$

We can verify that the definition makes sense by considering any observable O_S that acts on the Hilbert space of the system alone:

$$\langle O_S(t)\rangle = \mathrm{tr}\left[(O_S \otimes I_E)\chi(t)\right] = \mathrm{tr}_S\left[O\mathrm{tr}_E[\chi(t)]\right] = \mathrm{tr}_S[O\rho(t)]. \tag{3.9}$$

That is, the expectation value of any operator in our simplified state description is the same as the expectation value if we had considered the entire environment as well. This makes

sense, as a complete lack of knowledge of what occurs in the environment should not bias our measurement of the system of interest in any way.

Applying (3.8) to (3.6), one can obtain

$$\dot{\tilde{\rho}} = -\frac{1}{\hbar^2}\int_0^t dt' \mathrm{Tr}_E \left\{ \left[H_{SE}(t), [H_{SE}(t'), \tilde{\chi}(t')] \right] \right\}, \tag{3.10}$$

3.2.3 The Master Equation

In order to simplify the equation further, we need to introduce two significant approximations.

Born Approximation

When an object is put in contact with a large environment of a different temperature, one can assume that the overall temperature of the environment will not change significantly. That is, we assume the environment is large enough that it is not significantly affected by coupling with the system. This assumption is one that often still holds on the quantum scale, and thus we assume that $E(t) = E(0)$. In addition, we limit ourselves to the case where the coupling is weak, so that

$$\tilde{\chi}(t) = \tilde{\rho}(t)E(0) + O(H_{SE}). \tag{3.11}$$

That is, we ignore all terms of higher order in H_{SE}.

Markov Approximation

Suppose now that the temperature of the object varies drastically with respect to time. One would expect that such fluctuations will cause the environment to deviate from thermal equilibrium. That is, the state of the reservoir will be dependent on the entire history of thermal fluctuations from the object itself. Through the interaction between object and reservoir, the past history will again influence the future evolution of the object we are interested in.

However, provided the reservoir is sufficiently large, these correlations should die away quickly, and one can potentially assume that the future evolution of the system is governed only by the present state of the system, allowing us to replace $\tilde{\rho}(t')$ with $\tilde{\rho}(t)$ within the integral.

Together, these approximations[1] give the general master equation

$$\dot{\tilde{\rho}}(t) = -\frac{1}{\hbar^2}\int_0^t dt' \mathrm{Tr}_R \left\{ \left[H_{SE}(t), [H_{SE}(t'), \tilde{\rho}(t)E(0)] \right] \right\} = \mathcal{L}\tilde{\rho}. \tag{3.12}$$

The resulting equation now describes the evolution of our system in terms of the state of the system alone. That is, we have successfully avoided the requirement to have a specific

[1] The validity of the Born and Markov approximations are discussed in detail in works done by Haake [13] and Carmichael [12].

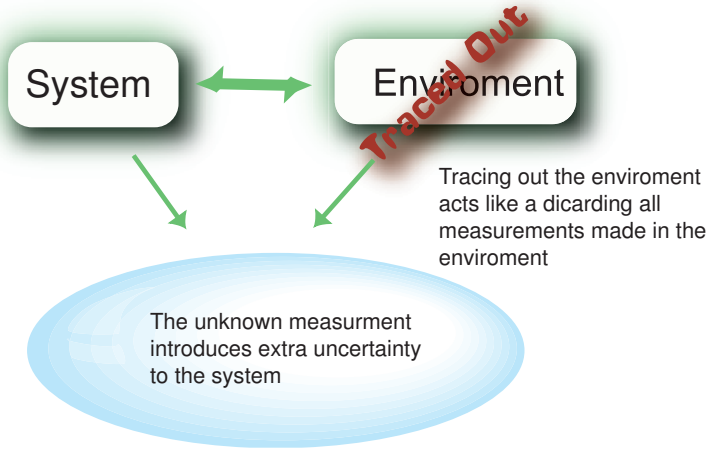

Figure 3.1: Information loss from the Master Equation.

dynamical description of our environment, while still able to incorporate the coupling H_{SE} into our model.

3.2.4 Missing Information

While the master equation is excellent at predicting the expectation value of physically observable quantities, it did so with a significant sacrifice. Recall that the derivation of the master equation required the process of a partial trace, it essentially assumes that we have no knowledge of any measurements made in the environment that could potentially give us better knowledge of the state of our system.

Essentially, this is no different from the process in which we defined entanglement. Here, Alice is now the system of interest, and Bob is the environment. We, as Alice did, have traced out the environment, which essentially equates to disregarding whatever measurement Bob has made there.

If there were no interactions, such that

$$H = H_S + H_E, \tag{3.13}$$

then, provided the state of system and environment could be factorized at $t = 0$, it will remain factorized for all time. And so, as the system and the environment are not entangled, the action of the partial trace will not cause us to lose any information about our system.

However, if there does exist an interaction H_{SE}, the action of H_{SE} on a factorized state will, in general, entangle system and environment. Let the amount of entanglement be denoted by $\mathcal{E}$ Bell states.

When we perform the partial trace, we will introduce exactly $\mathcal{E}$ bits of uncertainty into our system. Essentially, the price that we have paid to write an explicit dynamical equation

of the system, that is governed only by the system state, is information. That is, we have introduced extra uncertainty into our system (See Figure 3.1).

This extra uncertainty manifests itself in the use of density operators rather than pure states. That is, even if the initial state of our system is pure, the system will be described by a mixed state at general time t. We have lost information about the exact quantum state of the system.

3.3 Entanglement of Mixed States

With a mathematical description of open systems in place, we can return to our goal of formulating a description of how entanglement evolves in such systems. In Chapter 2, a complete theory of entanglement for pure states was formulated, but in light of the fact that the state of an open system is almost never pure, we will need to make generalizations.

3.3.1 Entanglement with Classical Probability

Consider two watermelons, one ripe and the other not ripe. We can denote the state 'ripe' by $|1\rangle$ and non-ripe by $|0\rangle$. Suppose now that Alice and Bob each takes one of these watermelons, then it is clear that if Bob cuts his open and finds it to be ripe, then he would automatically know that Alice's is non-ripe and vice versa.

Here, the two watermelons are correlated probabilistically, but only in a classical sense. That is, the state of our system is either $|00\rangle$ or $|11\rangle$, and the uncertainty that we have comes from the lack of knowledge, rather than any intrinsic quantum superposition of the system. In particular, the composite system is a mixed state, with uncertainty of 1 bit.

Suppose we were to discount Bob's subsystem; we lose no more knowledge than what we have already. Thus, it should be clear that such a system contains no entanglement. Yet, if we applied the formulae for entanglement given in (1), we would get a value of 1 for entanglement. It is clear that our definition for entanglement is only valid for pure states.

We can generalize our notion of entanglement to mixed states using the standard methods of probability theory, for the extra uncertainty is classical. That is, we an average of all possibilities, weighted according to its classical probability of occurrence.

Definition 6 (Entanglement of a Ensemble of Pure States) *Given a quantum ensemble of a composite system $A \otimes B$, such that the state of the system is $|\phi_i\rangle$ with probability p_i. The Von Neumann entanglement is defined by the classical expectation value of the entanglement*

$$E = \sum_i p_i E(|\phi_i\rangle), \tag{3.14}$$

where $E(|\phi_i\rangle)$ is the entanglement for the pure state $|\phi_i\rangle$

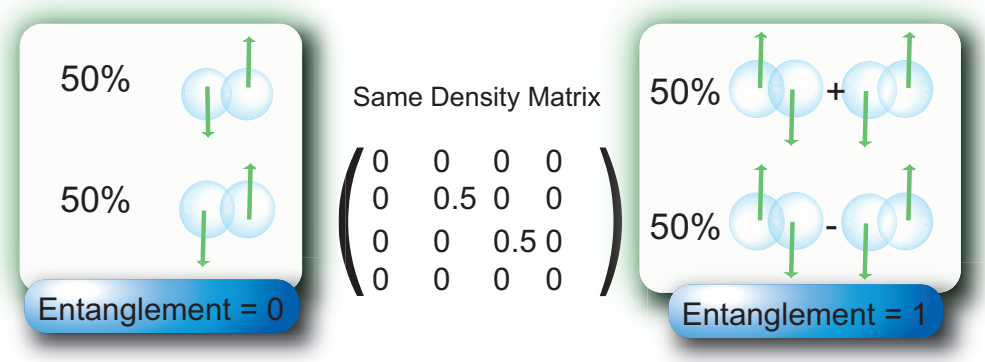

Figure 3.2: The same density operator can have multiple pure state unravellings, each of which has a different expected value of Von Neumann entanglement.

The definition makes sense. Give me two objects locked up in identical boxes, tell me that they both have entanglement of 0. Suppose I randomly pick a box, of course I will tell you that it has entanglement 0. Thus the entanglement of the situation mentioned above is 0, in correspondence with intuition.

3.3.2 Complications of Impurity

While Definition 6 is a perfectly valid measure of entanglement, it cannot be applied to many practical situations. The difficulty comes from the fact that our definition of entanglement is nonlinear in ρ, which implies that we cannot calculate the entanglement of an ensemble directly from the density operator, but must first decompose it into a probability distribution of pure states.

Let us return to our example where Alice and Bob each shared a watermelon. Prior to any measurements, the state of our system is already mixed, with density operator

$$\rho = \frac{1}{2}\left(|0\rangle|1\rangle\langle 0|\langle 1| + |1\rangle|0\rangle\langle 1|\langle 0|\right). \tag{3.15}$$

However, this density operator can also be rewritten as

$$\rho = \frac{1}{2}\left(|\phi_+\rangle\langle\phi_+| + |\phi_-\rangle\langle\phi_-|\right), \tag{3.16}$$

where

$$|\phi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle|0\rangle \pm |0\rangle|1\rangle\right) \tag{3.17}$$

are fully entangled Bell States (See Figure 3.2).

That is, the same density operator can be interpreted as the probabilistic average of two fully entangled Bell States, whereby, the Von Neumann entanglement would be evaluated as 1.

What went wrong here?

Proposition 3 (Non-Uniqueness of Entanglement for Density Operators) *The pure state decomposition (unravelling) of a density operator is not unique. Furthermore, different decompositions, in general, give rise to different values of entanglement.*

That is, there is no clearly defined measure of entanglement for density operators as there was for pure states. Or more accurately, the conversion from a probabilistic ensemble to a density operator is not one-to-one and thus we have lost the information required to uniquely define the entanglement.

3.3.3 Entanglement Measures for Non-Pure States

There are numerous proposals for quantifying entanglement for non-pure states which attempt to circumvent this problem. These roughly fall into a two categories

1. Measures that have a distinct physical interpretation, but are difficult or even impossible to compute.
2. Measures that have little physical interpretation but can be evaluated.

In this section, we will take a brief survey of the former category, as the latter is used for approximately comparative purposes, but does not really shed light on the physics.[2]

Physical Measures

When looking at how we can measure the entanglement of a mixed state, two very obvious possibilities come to mind. In the case pure states, it was shown that one could transform between n copies of an give state $|\phi\rangle$, and $nE|\phi\rangle$ Bell states using LOCC (Local Operations and Classical Communication). When we generalize this to mixed states, we can no longer assume this to be true. However, one could ask questions about the two operations that make up the statement. What is the minimum number of Bell states required to create a state with density operator ρ; and given a state ρ, how many Bell states can one be guaranteed to extract? This leaves us with two measures for mixed states with strong physical interpretations [14] [20].

Definition 7 (Entanglement of Formation) *Given a density matrix ρ, the Entanglement of Formation is given by*

$$E_F(\rho) = Min\left\{\sum_i p_i E(|\phi_i\rangle)\right\} \tag{3.18}$$

[2] A good example of a computable measure of entanglement with no clear physical interpretation is the Negativity.[18]

where the minimum is over all probabilistic decompositions of ρ.

Physically, this measure makes some sense. It asks, given a density operator, what is the minimum resource required to form the operator?

From the perspective of practical applications, it is more important to consider how much entanglement can be extracted from a general mixed state than the amount used to create it. This motivates the definition of the Entanglement of Distillation

Definition 8 (Entanglement of Distillation) *Given a state ρ, the Entanglement of Distillation $E_D(\rho)$ is the number of Bell states that can be purified per copy of ρ.*

Suppose that Alice and Bob were attempting to communicate over a noisy channel, such that the entangled states that they share must be described by a mixed state, given by a density operator ρ. We can guarantee that Alice and Bob are able to extract $E_D(\rho)$ Bell states from the mixture that can be used, for example, to teleport information [20]. Thus, Entanglement of Distillation is a very important point of consideration in quantum communication and error correction protocols.

The two measures above arise directly from physical arguments, and satisfy a set of postulates that we intuitively believe and any measure for entanglement should satisfy. Such a set of postulates was proposed by Borodecki [25].

1. **Non-Negativity:** $E(\rho) \geq 0$
2. **Consistency:** $E(\rho) = 0$ for a state ρ that has a separable expansion
3. **Normalization:** $E(|\phi\rangle\langle\phi|) = 1$ for Bell states $|\phi\rangle$
4. **Subadditivity:** $E(\rho^{\otimes n}) = nE(\rho)$
5. **Monotonicity:** $E(\rho)$ does not increase from LOCC (Local Operations and Classical Communications)
6. **Continuity:** If $\langle\phi^{\otimes n}|\rho_n|\phi^{\otimes n}\rangle \to 1$ as $n \to \infty$, then $\frac{1}{n}|E(|\phi\rangle^{\otimes n}) - E(\rho_n)| \to 0$

Most of these conditions are obvious. The first three guarantee agreement with our general notions of entanglement and the fundamental measure; subadditivity essentially reasons that the entanglement of n identical systems should be n times that of a single system; and monotonicity formally states our intuition that we cannot increase Entanglement using operations that contain no quantum correlations. Finally, continuity guarantees that our measure converges to expected results in the limit that the density operator tends towards a pure state. It can be verified that both $E_F(\rho)$ and $E_D(\rho)$ satisfy the above conditions.

In fact, we can define an entire class of physical entanglement measures which satisfy the above postulates. For example, the Quantum Relative Entropy first proposed by Vedral [17].

Definition 9 (Quantum Relative Entropy) *Given a state ρ, the Quantum Relative Entropy is defined by*

$$E_R(\rho) = Min_{\sigma\in\mathcal{D}} \{ Tr[\rho \lg \rho - \rho \lg \sigma] \}, \tag{3.19}$$

where $\mathcal{D}$ is the set of all density operators with a separable decomposition.

We note that the term

$$\mathrm{Tr}\,[\rho \lg \rho - \rho \lg \sigma] \tag{3.20}$$

is a generalization of the classical relative entropy of two distributions p_k and q_k,

$$d = \sum_k p_k \lg \left(\frac{p_k}{q_k} \right), \tag{3.21}$$

which is essentially a distance measure between the two classical distributions [26]. Thus, the quantum relative entropy is a measure of the closest distance between ρ and any separable state.

As all entanglement measures mentioned satisfy monotonicity, it makes sense that you cannot extract more entanglement out of a system than you put in. In fact, this was formulated into a fundamental theorem for physical measures of entanglement [25].

Theorem 4 (Limits of Entanglement Measures) *For any physical entanglement measure E that satisfy the above postulates and any density operator ρ,*

$$E_D(\rho) \leq E(\rho) \leq E_F(\rho). \tag{3.22}$$

That is to say, if we were to formulate a measure of entanglement, using the units of Bell states, that makes physical sense, then it is bounded above by the Entanglement of Formation and below by the Entanglement of Distillation. For pure states $E_D(\rho) = E_F(\rho)$, which demonstrates that there is a unique entanglement measure for pure states, namely the Von Neumann Entanglement.

With a theory of mixed state entanglement firmly at hand, we can attempt to apply it to our initial example of spin half particles shared between Alice and Bob, with density operator

$$\rho = \frac{1}{2} \left(|0\rangle|1\rangle\langle 0|\langle 1| + |1\rangle|0\rangle\langle 1|\langle 0| \right). \tag{3.23}$$

It is clear that no entanglement is required to form such a state (for we can achieve the same operator assuming the spin half particles were classical watermelons). Theorem 4 can then be immediately applied to conclude that no physical measure of entanglement will give a non-zero value for the entanglement of the above state; yet we have already demonstrated that such a state could have been expanded in terms of perfectly entangled states.

In a certain light, the results make sense. Without better knowledge of how the density operator was formed, physical measures that conserve entanglement are forced to consider the unravelling of least entanglement. Otherwise, monoticity would be violated.[3]

[3]For example, a person could use two watermelons to generate 'entanglement'.

In our example, $S(\rho) = 1$. That is, our system started off with an uncertainty for one bit. It should then be of no surprise that different unravellings of the density matrix could give entanglement measures of up to one bell State in difference.

After all, how does one measure a property of a given object precisely when that object is not exactly defined in the first place?

3.3.4 The Peculiarity of Entanglement

Classically, this result should be considered exceedingly strange. Consider two batteries of different brands, but with an identical amount of energy. A lab technician, for reasons beyond the scope of this thesis, decides to paint both batteries bright blue. When you ask, as you do each day, for a battery to power your oscilloscope, a bright blue battery lands in your hands. You look at it, shrug, and place it in the oscilloscope. Everything works.

And why shouldn't it?

Why would the brand matter, given that you know they all contain energy, the resource that you need?

Yet, in a quantum world, when different brands create different states, all featuring the same entanglement, it will matter. For a mixture of two bell states can have zero entanglement. Or more explicitly, your lack of knowledge prevents you access to the resource, and without a way to access it, can we really argue that it exists? A natural way to avoid this situation is to keep that knowledge within our model, to 'peel away the paint'.

3.4 Quantum Trajectory Theory

The intrinsic problem in quantifying a mixed state is the multiplicity of pure state expansions. This lack of information comes about from the fact that the density operator is not really a fundamental physical state, but can be considered a collection of different pure state ensembles.

Asking the exact value of entanglement for a density operator is like asking the exact mass of a watermelon. Just as each individual watermelon has a different mass, each expansion of a density operator may have differing values of entanglement. It becomes quickly apparent that a definite measurement of entanglement for such mixed states cannot really be defined.

An alternative approach is to return to the source of the uncertainty, which made entanglement so difficult to characterize. Is there a way to include more information from the environment than a partial trace? We will first give a rough outline of the idea, and then provide a more rigorous treatment of the method.

3.4.1 Unravelling the Density Operator

Consider a system that is coupled with the environment. We know from Chapter 2 that by taking the partial trace with respect to the system, the purity of the system was destroyed. In particular, a mixed state with many possible decompositions resulted.

Suppose that at some time t, our system and the environment were entangled in the form

$$|\phi\rangle = \sum_{i,j} c_{i,j}|\phi_i\rangle_S|\phi_j\rangle_E. \tag{3.24}$$

A partial trace with respect to E would immediately reduce S to a mixed state. However, we could instead hypothesize that an observer makes a measurement in the environment with respect to the basis $|\phi_j\rangle_E$. A measurement result will collapse the entangled state into

$$|\phi\rangle = \mathcal{N}\left(\sum_i c_{i,j}|\phi_i\rangle_S\right)|\phi_j\rangle_E \tag{3.25}$$

with probability $p_j = \sum_i |c_{i,j}|^2$. Since this resulting state is now a simple product state of system and environment, the environment can be traced out to get a pure state for the system.

In this formulation, we have included a well defined measurement done by an observer on the environment. The action of this measurement will cause a collapse within in the system of interest, and define a unique unravelling the density operator. This way, the entanglement of our system remains exact.

In order to characterize the evolution of the system, we will need to consider all possible measurement results by the observer at all continuous time intervals t. This results in an infinite number of quantum trajectories whose ensemble average gives the density matrix whose dynamics are governed by the Master Equation (3.12). This is the key idea of the Quantum Trajectories formalism proposed by Carmichael [21].

By including the hypothetical continuous measurement made on the environment, we are able to produce an unravelling of the density operator that has a direct physical interpretation. In such an unravelling of $\rho = \sum_i p_i|\phi_i\rangle\langle\phi_i|$, p_i denotes the probabilities of getting the sequence of measurements that results in the system being collapsed to the pure state $|\phi_i\rangle$.

It is then natural to propose a measure of entanglement for this open system by [22]

$$E_U = \sum_i p_i E(|\phi_i\rangle), \tag{3.26}$$

where p_i is now the probability of recording a particular set of measurements, and $|\phi_i\rangle$ the resulting state given such a recording.

Effectively, the measurement history that becomes a key aspect to the dynamics of the system. It is the tool that we can use to define a unique unravelling. With this information, we can uniquely determine and extract the entanglement from an open system.

3.4.2 Mathematical Formulation

In order to apply the conceptual theory mentioned above, we need to formulate the procedure in terms of the system of interest alone. How does one mathematically describe measurements made by the outside observer, and predict how they will affect our system of interest? To answer this question, we will outline the work done by Carmichael **??**.

Consider a general system of interest, governed by a master equation for the reduced density operator ρ.

$$\dot{\rho} = \mathcal{L}\rho. \tag{3.27}$$

In general, we can rewrite

$$\mathcal{L} = (\mathcal{L} - \mathcal{S}) + \mathcal{S} = \mathcal{L}_0 + \mathcal{S}, \tag{3.28}$$

whereby the system has the formal solution

$$\rho(t) = e^{\mathcal{L}_0 + \mathcal{S}}\rho(0). \tag{3.29}$$

Applying the Dyson expansion, we obtain

$$\rho(t) = \sum_{m=0}^{\infty} \left(\int_0^t dt_m \int_0^{t_m} dt_{m-1} \ldots \int_0^{t_2} dt_1 e^{\mathcal{L}_0(t-t_m)} \mathcal{S} e^{\mathcal{L}_0(t_m - t_{m-1})} \mathcal{S} \ldots e^{\mathcal{L}_0 t_1} \rho_0 \right). \tag{3.30}$$

While this equation may look intimidating, we can decompose it into an infinite summation of terms that resemble

$$e^{\mathcal{L}_0(t-t_m)} \mathcal{S} e^{\mathcal{L}_0(t_m - t_{m_1})} \mathcal{S} \ldots e^{\mathcal{L}_0 t_1} \rho_0. \tag{3.31}$$

Here, the operators $e^{\mathcal{L}_0(t_m - t_{m-1})}$ represent continuous evolution for a time period of $t_m - t_{m-1}$, while $\mathcal{S}$ represents sudden 'jumps'. Thus, each term in the summation represents free evolution via a modified superoperator $\mathcal{L}_0$ coupled with discontinuous collapses at times $t_1, t_2, \ldots, t_m$. We define this as one quantum trajectory. Equation (3.30) is then simply a summation over all possible trajectories.

While the above is true for all choices of $\mathcal{L}_0 + \mathcal{S} = \mathcal{L}$, there are certain specific choices that make physical sense. In particular, for many open systems, such as a driven atom undergoing resonance fluorescence, and spontaneous emission [21], the master equation can be expressed in the form

$$\begin{aligned} \dot{\rho} &= (\mathcal{L}_0 + S)\rho, \\ \mathcal{L}_0\rho &= \frac{1}{i\hbar}[H, \rho] - (C^\dagger C\rho + \rho C^\dagger C), \\ S\rho &= C\rho C^\dagger. \end{aligned} \tag{3.32}$$

where C, defined as the collapse operator, is particular to the system. Generally C represents the collapse of the system after a measurement result.

In this formalism, we can let $\rho = |\phi\rangle\langle\phi|$ and reduce the evolution of the system to that of a pure state evolving under an effective Hamilton H_{eff}:

$$\frac{d|\phi\rangle}{dt} = \frac{1}{i\hbar} H_{eff}|\phi\rangle, \qquad H_{eff} = H - \frac{i}{2}C^\dagger C \tag{3.33}$$

$|\phi\rangle$ can be interpreted as the pure state of the system, which evolves under an effective Hamiltonian, H_{eff}. Thus, the free evolution period of each trajectory behaves similarly to that of the evolution a typical wavefunction in Schrödinger Mechanics.

The open nature of the system is reflected by the fact that H_{eff} is in general not hermitian, and thus the norm of $|\phi\rangle$ is not conserved and diminishes over time. In fact

$$P_w(t) = ||\phi(t)||^2 \tag{3.34}$$

denotes the *no-collapse probability*, that is, the probability that the system will evolve under H without a quantum collapse. This is a consistent generalization from closed systems, where the norm of $|\phi\rangle$ is conserved, and thus there is no chance of a collapse. At any time t, we define

$$|\phi_c(t)\rangle = \frac{|\phi(t)\rangle}{||\phi(t)||} \tag{3.35}$$

as the conditioned wavefunction, which represents the true state of the system.

The sudden collapses defined by the superoperator $\mathcal{S}$ can now be interpreted as collapses caused by measurements made on the the environment. Thus, the less coupling between system and environment, the slower $||\phi(k)||$ decays, and the less likely it is for a quantum collapse.

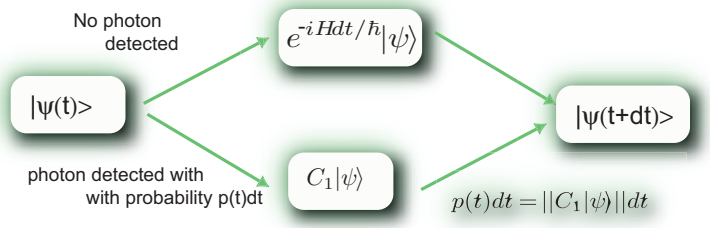

Figure 3.3: The evolutionary process of a single quantum trajectory.

During the time interval $(t, t+dt)$, there is a finite chance that the system undergoes a sudden quantum collapse, which corresponds to a measurement made in the environment, such as the detection of a photon. The probability of collapse is given by

$$p_c(t)dt = \langle\phi(t)|C^\dagger C|\phi(t)\rangle dt, \tag{3.36}$$

whereby the system is collapsed to the state

$$\frac{C|\phi(t)\rangle}{\langle\phi(t)|C^\dagger C|\phi(t)\rangle}. \tag{3.37}$$

This stochastic process is summarized in Figure 3.3.

Each of of these stochastic trajectories can be interpreted as an actual realization of a physical experiment, provided the appropriate measurements were made on the environment. The density operator is simply the stochastic average over all trajectories [21].

3.4.3 The Resulting Formalism

> "Do you think you understand the basic ideas of Quantum Mechanics"
> "Ah! Well, what do we mean by 'to understand' in the context of Quantum Mechanics?"
> -*Anonymous conversation between supervisor and student*

Quantum trajectory theory, in effect, decomposes the master equation into an infinite number of possible physical realizations, which we term quantum trajectories. Each trajectory can correspond to the results of one run of an idealized experiment, where each individual collapse signifies a collapse of wavefunction due to some external measurement, such as the detection of a photon emission.

The most interesting thing of note is that trajectory theory explicitly demonstrates that the evolution of an open quantum system is dependent on the measurement history made within the environment. In a classical sense, this should be regarded as absolutely weird, for when you detect that photon which has bounced off the watermelon in mid-flight, you do not expect the action of detection to affect the watermelon itself in any way. After all, the interaction between the watermelon and the photon was in the past!

This paradox is no more than a more elaborate version of the quantum measurement problem. Any interactions between system and environment will in general, generate entanglement between the two, and thus a measurement on the environment would collapse the wavefunction of the system. The system's evolution is then obviously dependent on the results of your measurement and what you choose to measure.

This also leads us to the interesting idea of using specific measurements to increase and enhance the entanglement of a particular open system [22]. Indeed, this method of conditional preparation via continuous measurements has already been proposed [24].

In Chapter 5, we will apply this formalism to decipher the behavior of entanglement in a practical physical system, and demonstrate the applications made above. In fact, we will see that it is possible to obtain a good picture of how entanglement evolves by considering the entanglement of each individual trajectory alone.

Chapter 4

The Cascaded Cavity System

"Example isn't another way to teach, it is the only way to teach. "
-*Albert Einstein*

4.1 Introduction

We have seen that quantum trajectories are a promising technique for studying the entanglement of open systems. In this formulation, we compensate for the information lost to the environment by incorporating measurements upon the environment into our model, thereby alleviating the degeneracy of pure state expansions for the system density operator.

We will now apply the proposed method to the analysis of an open system of practical interest. In 2003, Clark and Parkins [27] demonstrated that a cascaded cavity system is capable of generating perfectly entangled steady states in the limit of ideal cavity-cavity coupling. Essentially it involves two high finesse optical cavities arranged in a cascaded configuration with unidirectional coupling. Within each cavity is a tightly confined atom with five electronic levels, two of these levels are ground states that represent an effective qubit.

In this chapter, we will outline a more general version of such a system, where each cavity contains not one, but an arbitrary number of atoms. This allows us to not only consider the case of qubit entanglement, but also entanglement between two atomic clouds. The goal is to develop a basic mathematical model for the system, in terms of master equations, which can then be used to create a quantum trajectory formalism for the system.

Figure 4.1 demonstrates a simplified schematic of the cascaded cavity system. n five level atoms are constrained in each of the two optical cavities, where we denote the ground states by $|0\rangle$ and $|1\rangle$. There are also short lived excited states $|r\rangle$, $|s\rangle$ and $|t\rangle$.

Manipulations between states $|0\rangle$ and $|1\rangle$ are occur via the excited states $|r\rangle$ and $|s\rangle$. To minimize time spent in these excited states, independent laser fields that drive transitions

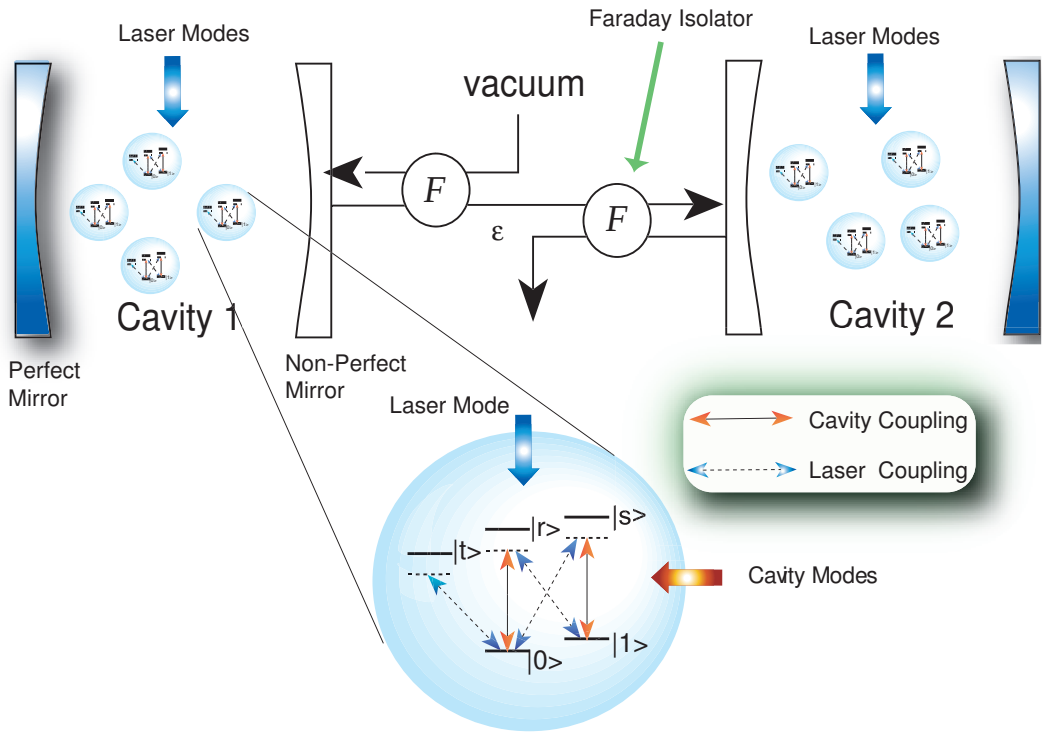

Figure 4.1: Schematic of the cascaded cavity system. Atoms in each cavity are driven by 3 independent detuned, laser fields corresponding to each of the three labeled transitions. Each cavity outputs the photon emissions caused by the coupling of the cavity modes.

$|0\rangle \leftrightarrow |s\rangle$ and $|1\rangle \leftrightarrow |r\rangle$ are highly detuned, as is the cavity mode which strongly couples to the transitions $|0\rangle \leftrightarrow |r\rangle$ and $|1\rangle \leftrightarrow |s\rangle$. In effect, a transition driven by a laser field from $|0\rangle \rightarrow |s\rangle$ is followed by an effectively instant induced transition down to $|1\rangle$ and vice versa. In this regime, we have an effective way to control transitions between $|0\rangle$ and $|1\rangle$, and we are able to disregard spontaneous emission from the excited states.

Secondly, we wish to construct the cavities in such a way that the time it takes for a photon to escape the cavity is much smaller than the average time a photon requires to be reabsorbed by the atom. This allows us to assume that all photons interact only once within the cavity, and that there is no chance for the cavity field to induce $|0\rangle \rightarrow |r\rangle$ and $|1\rangle \rightarrow |s\rangle$ transitions. The absence of such absorptions will enhance our control of the system. We call this the *Bad Cavity Limit*.

Finally, the extra energy level $|t\rangle$ is used for 'tuning'. Virtual transitions from $|0\rangle$ to $|t\rangle$ allows to fine tune the energy of $|0\rangle$ via the AC-Stark Shift.

The output of the first cavity is used to drive the second cavity, where input and output are separated by Faraday isolators. We can characterize the efficiency of this coupling by the parameter ϵ. The output of the second cavity can then be directly measured.

Clearly as this system has a visible output, it is an open system. Thus, the general state of

this system is mixed, and we will see that quantum trajectories can help us understand how the system evolves.

The first step, however, is to construct a mathematical model featuring only the states of interest. In this case, we are interested only in the entanglement between the ground states of the atoms within each cavity, and thus in this chapter, our aim is to write a master equation for the system in terms of these states alone.

4.2 The Single Cavity Sub-System

It is often the case that to analyze a large composite system, it is easiest to break it up into smaller components. The cascaded system is no different, and can be broken up into two modules, each of which consists of a single cavity. In this section, we will consider such a cavity in detail, and demonstrate the techniques that can be used to derive a master equation for such a sub-system in terms of only the ground states of each individual atom.

4.2.1 The System Hamiltonian

Figure 4.2 shows a detailed energy level diagram for each individual atom inside each cavity. Observe that

- Ω_r, Ω_s and Ω_t are the Rabi frequencies of the lasers driving transitions $|1\rangle \leftrightarrow |r\rangle$, $|0\rangle \leftrightarrow |s\rangle$ and $|0\rangle \leftrightarrow |t\rangle$, respectively.
- Each of the lasers also has frequencies ω_{L_r}, ω_{L_s} and ω_{L_t}, respectively.
- Δ_r, Δ_s and Δ_t are the detunings from the excited states $|r\rangle$, $|s\rangle$ and $|t\rangle$.
- g_r and g_s are the coupling strengths of the transitions $|0\rangle \leftrightarrow |r\rangle$ and $|1\rangle \leftrightarrow |s\rangle$ with respect to the cavity mode.
- Each energy state $|i\rangle$ $(i = 1, t, r, s)$ has a energy difference of ω_i with respect to $|0\rangle$
- ω is the frequency of the cavity mode.
- Exact Raman resonance is assumed, such that $\omega - \omega_{L_r} = \omega_1$, $\omega - \omega_{L_s} - \omega = \omega_1$

Note that for ease of analysis, we have defined our units such that $\hbar = 1$. We will also assume that all of the atoms within the cavity are completely identical, with identical coupling strength to the incident lasers and cavity modes, i.e: $g_{r,i} = g_r$, $g_{s,i} = g_s$, $\Omega_{r,i} = \Omega_r$, $\Omega_{s,i} = \Omega_s$ for $i = 1 \ldots n$.

Treating the incident lasers semi-classically, and quantizing the cavity mode, we use the formalism given for basic atom light interactions in Chapter 5 of [28] to directly write down a Hamiltonian for this system:

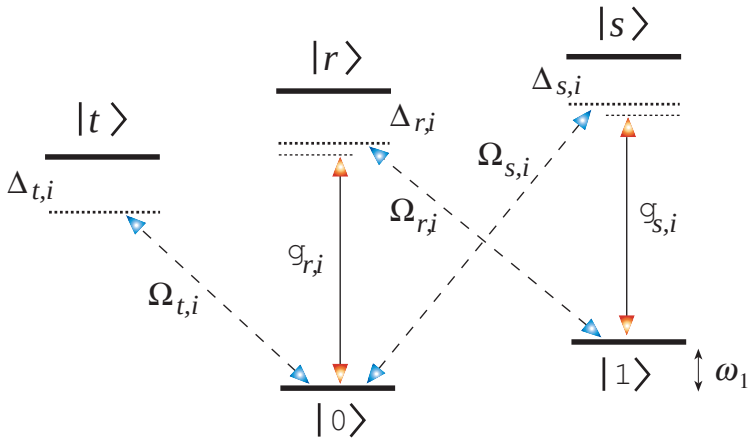

Figure 4.2: Level diagram of the 5-level atom.

$$H = \underbrace{H_{\text{cav}}}_{\text{Single Mode Cavity}} + \underbrace{H_{\text{atoms}}}_{\text{Atom}} + \underbrace{H_{\text{atom/lasers}}}_{\text{Atom-Laser Interaction}} + \underbrace{H_{\text{atom/cav}}}_{\text{Atom-Cavity Interaction}} \tag{4.1}$$

$$H_{\text{cav}} = \omega a^\dagger a \tag{4.2}$$

$$H_{atoms} = \sum_{i=1}^{n} (\omega_r |r\rangle\langle r|_i + \omega_s |s\rangle\langle s|_i + \omega_t |t\rangle\langle t|_i \ + \Delta_{01}|1\rangle\langle 1|) \tag{4.3}$$

$$H_{\text{atoms/lasers}} = \sum_{i=1}^{n} \left(\frac{\Omega_r}{2} e^{-i\omega_{Lr}t} |r\rangle\langle 1|_i + \frac{\Omega_s}{2} e^{-i\omega_{Ls}t} |s\rangle\langle 0|_i + \frac{\Omega_t}{2} e^{-i\omega_{Lt}t} |t\rangle\langle 0|_i + \text{H.c} \right) \tag{4.4}$$

$$H_{\text{atoms/cav}} = \left(\sum_{i=1}^{n} g_r |r\rangle\langle 0|_i a + g_s |s\rangle\langle 1|_i a + \text{H.c} \right) \tag{4.5}$$

The dynamics of the system is now given by the master equation

$$\dot{\chi} = \mathcal{L}\chi = -i[H, \chi] + \mathcal{L}_{\text{cav}}\chi, \tag{4.6}$$

where we have introduced $\mathcal{L}_{\text{cav}}$, the cavity Liouvillian, which describes the damping caused by light escaping the cavity. We note that is the term that couples with the environment, and thus causes our system to lose information.

As the main focus of this thesis is to describe the reasons for information loss and the analysis of entanglement evolution, we refer the reader to [12] for the derivation of

$$\mathcal{L}_{\text{cav}}\chi = 2a\chi a^\dagger - a^\dagger a\chi - \chi a^\dagger a. \tag{4.7}$$

Here χ is the density operator for the entire system, which consists of each ground and excited state for each of the n atoms within the cavity, as well as the cavity mode.

4.2.2 Switching to the Interaction Picture

We will first eliminate the explicit time dependence in the Master Equation (4.1) by transforming into the interaction picture. Here, we choose a frame of reference such that it does not rotate with respect to the excited states of the atom, the stationary cavity mode, or ground state $|1\rangle\langle 1|$. Explicitly, we let

$$\tilde{\chi} = e^{iH_0 t}\chi e^{-iH_0 t} \tag{4.8}$$

where

$$H_0 = \sum_{i=1}^{n} (\omega_{L_r} + \omega_1)|r\rangle\langle r|_i + \omega_{L_s}|s\rangle\langle s|_i + \omega_{L_t}|t\rangle\langle t|_i + \omega_1|1\rangle\langle 1|_i) + (\omega_{L_s} - \omega_1)a^\dagger a. \tag{4.9}$$

It follows from the condition of Raman resonance that the change of co-ordinates simplifies our master equation into the form

$$\dot{\tilde{\chi}} = -i[Q + Q^\dagger + Z, \tilde{\chi}] + \mathcal{L}_{\text{cav}}\tilde{\chi}, \tag{4.10}$$

where

$$\begin{aligned} Q &= \sum_{i=1}^{n} \mathbf{I}^{\otimes(i-1)} \otimes Q_i \otimes \mathbf{I}^{\otimes(n-i)}, && (4.11)\\ Q_i &= g_r a^\dagger|0\rangle\langle r|_i + g_s a^\dagger|1\rangle\langle s|_i + \frac{\Omega_r}{2}|r\rangle\langle 1|_i \\ &\quad + \frac{\Omega_s}{2}e^{i\phi}|s\rangle\langle 0|_i + \frac{\Omega_t}{2}|t\rangle\langle 0|_i, && (4.12) \end{aligned}$$

describes the couplings between each atom and the electromagnetic field and

$$\begin{aligned} Z_i &= -\Delta_r|r\rangle\langle r|_i - \Delta_s|s\rangle\langle s|_i - \Delta_t|t\rangle\langle t|_i, && (4.13)\\ Z &= \sum_{i=1}^{n} \mathbf{I}^{\otimes(i-1)} \otimes Z_i \otimes \mathbf{I}^{\otimes(n-i)}. && (4.14) \end{aligned}$$

Z represents the left over rotation due to detuning between the laser and atoms.

4.2.3 Splitting the Master Equation

Recall that we wish for each individual atom within each optical cavity to function as a qubit. Thus, we need to increase the laser detunings to a point such that the time each atom spends in its excited states is negligible.

Assumption 1 (Large Detuning) *We assume that the lasers are highly detuned from the excited states. Explicitly,*

$$|\Delta_j| \gg \Omega_j, g_r, g_s, \kappa \qquad j = r, s, t. \tag{4.15}$$

Consequently, if p is directly proportional to the probability a given atom is in an excited state, then we are able to drop all terms quadratic in p. This approximation allows us to use projection operators to eliminate the need to explicitly include excited states within our model:

$$\begin{aligned} P_i^+ &= |r\rangle\langle r|_i + |s\rangle\langle s|_i + |t\rangle\langle t|_i & (4.16)\\ P_i^- &= |1\rangle\langle 1|_i + |0\rangle\langle 0|_i & (4.17) \end{aligned}$$

P_i^+ and P_i^- are used to project the i^{th} atom to the excited and ground states, respectively.

We can then form a global projection by taking various combinations of their tensor product

$$P^{s_1 s_2 s_3 \ldots s_n} = P_1^{s1} \otimes P_2^{s2} \otimes \ldots \otimes P_n^{s_n}, \qquad s_i \in \{+, -\}. \tag{4.18}$$

We can use (4.18) to project our system of atoms into the exact configuration of ground and excited states that we wish. In practice, it is useful to introduce a more compact notation. Let

$$P^{\mathbf{s}(A)} = P^{s_1 s_2 s_3 \ldots s_n} \tag{4.19}$$

such that $s_k = + \quad \forall i \in A$ and $s_k = -$ otherwise.

Partial Master Equations

We can split (4.10) into 2^{2n} coupled differential equations by defining

$$\tilde{\chi}^{\mathbf{s}(A)\mathbf{s}(B)} = P^{\mathbf{s}(A)} \tilde{\chi} P^{\mathbf{s}(B)}. \tag{4.20}$$

Fortunately, it is unnecessary to deal with all 2^{2n} terms. As any term with more than two atoms in their excited states will be quadratic in p and can therefore be neglected. Consequently

$$\tilde{\chi}^{\mathbf{s}(A)} = 0 \qquad \forall \mathbf{s}(A): \quad |A| > 2 \tag{4.21}$$

At this point, it is useful to state some simple commutation relations between P and our system of interest:

$$\begin{aligned} P^{--\cdots--} Q^\dagger &= P^{\mathbf{s}(\emptyset)} Q^\dagger = \mathbf{0} & (4.22)\\ P^{\mathbf{s}(\{k\})} Q^\dagger &= \left(\mathbf{I}^{\otimes(k-1)} \otimes Q_k^\dagger \otimes \mathbf{I}^{\otimes(n-k)} \right) P^{\mathbf{s}(\emptyset)} & (4.23) \end{aligned}$$

and for general $\mathbf{s}(A)$

$$P^{\mathbf{s}}(A)Q^{\dagger} = \sum_{i\in A}\left(\mathbf{I}^{\otimes(i-1)}\otimes Q_i^{\dagger}\otimes\mathbf{I}^{\otimes(n-i)}\right)P^{\mathbf{s}(A-\{i\})}. \tag{4.24}$$

By similar reasoning,

$$P^{\mathbf{s}}(A)Q = \sum_{i\notin A}\left(\mathbf{I}^{\otimes(i-1)}\otimes Q_i\otimes\mathbf{I}^{\otimes(n-i)}\right)P^{\mathbf{s}(A+\{i\})}. \tag{4.25}$$

In addition

$$\begin{aligned} P^{\mathbf{s}(\emptyset)}Z &= \mathbf{0} && (4.26)\\ P^{\mathbf{s}(\{k\})}Z &= \left(\mathbf{I}^{\otimes(k-1)}\otimes Z_k\otimes\mathbf{I}^{\otimes(n-k)}\right)P^{\mathbf{s}(\{k\})}. && (4.27)\end{aligned}$$

Applying these relations to the reduced master equation (4.10), we can write down the dynamics for all significant density operator components:

$$\begin{aligned} \frac{d}{dt}\tilde{\chi}^{\mathbf{s}(\emptyset)\mathbf{s}(\emptyset)} &= -iP^{\mathbf{s}(\emptyset)}[Q+Q^{\dagger}+Z,\tilde{\chi}]P^{\mathbf{s}(\emptyset)}\\ &= -iP^{\mathbf{s}(\emptyset)}Q\tilde{\chi}P^{\mathbf{s}(\emptyset)}+\text{H.c}\\ &= -i\sum_{k=1}^{n}\left(\mathbf{I}^{\otimes(k-1)}\otimes Q_k\otimes\mathbf{I}^{\otimes(n-k)}\right)\tilde{\chi}^{\mathbf{s}(\{k\})\mathbf{s}(\emptyset)}+\text{H.c} && (4.28)\\ \frac{d}{dt}\tilde{\chi}^{\mathbf{s}(\{k\})\mathbf{s}(\emptyset)} &= -i\left(\mathbf{I}^{\otimes(k-1)}\otimes Z_k\otimes\mathbf{I}^{\otimes(n-k)}\right)\tilde{\chi}^{\mathbf{s}(\{k\})\mathbf{s}(\emptyset)}\\ &\quad -i\left(\mathbf{I}^{\otimes(k-1)}\otimes Q_k^{\dagger}\otimes\mathbf{I}^{\otimes(n-k)}\right)\tilde{\chi}^{\mathbf{s}(\emptyset)\mathbf{s}(\emptyset)} && (4.29)\end{aligned}$$

Note that all other partial density operators have been ignored, in correspondence to the large detuning assumption.

Adiabatic Elimination Process

By *Assumption 1*, we can assume that the coherence terms $\tilde{\chi}^{\mathbf{s}(\{k\})\mathbf{s}(\emptyset)}$ follow adiabatically the population coherences $\tilde{\chi}^{\mathbf{s}(\emptyset)\mathbf{s}(\emptyset)}$ in the ground states. That is, we can set

$$\frac{d}{dt}\tilde{\chi}^{\mathbf{s}(\{k\})\mathbf{s}(\emptyset)} = 0, \tag{4.30}$$

and solve for $\tilde{\chi}^{\mathbf{s}(\{k\})\mathbf{s}(\emptyset)}$ in terms of $\tilde{\chi}^{\mathbf{s}(\emptyset)\mathbf{s}(\emptyset)}$. We will also define $\rho = \tilde{\chi}^{\mathbf{s}(\emptyset)\mathbf{s}(\emptyset)}$ as the reduced density operator for the ground states of the atoms. Applying the condition (4.30) to (4.29) immediately gives

$$\tilde{\chi}^{\mathbf{s}(\{k\})\mathbf{s}(\emptyset)} = -\left(\mathbf{I}^{\otimes(k-1)} \otimes Z_k^{-1} Q_k^\dagger \otimes \mathbf{I}^{\otimes(n-k)}\right) \rho, \tag{4.31}$$

which can then be substituted into (4.28) to obtain a master equation for ρ.

$$\frac{d\rho}{dt} = -i[H_g, \rho] \tag{4.32}$$

with

$$H_g = -\sum_{k=1}^{n} \left(\mathbf{I}^{\otimes(k-1)} \otimes Q_k Z_k^{-1} Q_k^\dagger \otimes \mathbf{I}^{\otimes(n-k)}\right). \tag{4.33}$$

This model allows us to describe the evolution of the cascaded system in terms of the ground states of the atoms and the cavity modes only.

The Reduced Master Equation

An explicit form of H_g can be written by applying the specific forms of Q_i (4.11) and Z_i (4.13). Define

$$\eta_{r,s} = \frac{g_{r,s}^2}{\Delta_{r,s}}, \qquad \beta_{r,s} = \frac{g_{r,s}\Omega_{r,s}}{\Delta_{r,s}}. \tag{4.34}$$

We find

$$\begin{aligned} -Q_i Z_i^{-1} Q_i^\dagger \;=\; & a^\dagger a \left(\eta_r |0\rangle\langle 0| + \eta_s |1\rangle\langle 1|_i\right) \\ & + \beta_r \left(a^\dagger |0\rangle\langle 1|_i + a|1\rangle\langle 0|_i\right) + \beta_s \left(a^\dagger |1\rangle\langle 0|_i e^{i\phi} + a|0\rangle\langle 1|_i e^{-i\phi}\right) \\ & + \frac{\Omega_r^2}{4\Delta_r}|1\rangle\langle 1|_i + \frac{\Omega_s^2}{4\Delta_s}|0\rangle\langle 0|_i + \frac{\Omega_T^2}{4\Delta_T}|0\rangle\langle 0|_i. \end{aligned} \tag{4.35}$$

Thus we can rewrite the master equation for the ground states of all atoms within the cavity, including the cavity mode as

$$\frac{d\rho}{dt} = -i[H_g, \rho] + \mathcal{L}_{\text{cav}}\rho, \tag{4.36}$$

with Hamiltonian

$$\begin{aligned} H_g &= \sum_{i=1}^{n} a^\dagger a \left(\eta_r \sigma_i^- \sigma_i^+ + \eta_s \sigma_i^+ \sigma_i^-\right) \\ &+ \sum_{i=1}^{n} \left[\beta_r \left(a^\dagger \sigma_i^- + a\sigma_i^+\right) + \beta_s \left(a^\dagger \sigma_i^+ e^{i\phi} + a\sigma_i^- e^{-i\phi}\right)\right] \\ &+ \sum_{i=1}^{n} \left(\frac{\Omega_r^2}{4\Delta_r}\sigma_i^+\sigma_i^- + \frac{\Omega_s^2}{4\Delta_s}\sigma_i^-\sigma_i^+ + \frac{\Omega_t^2}{4\Delta_t}\sigma_-^+\sigma_+^-\right) \end{aligned} \tag{4.37}$$

where $\sigma_i^- = |0\rangle\langle 1|_i$ is the lowering operator for the i^{th} atom.

Recall that the state $|t\rangle$ was introduced as an auxiliary state whose parameters can be tuned to our advantage. Suppose we choose its coupling strengths such that the equality

$$-\frac{\Omega_r^2}{4\Delta_r} + \frac{\Omega_s^2}{4\Delta_s} + \frac{\Omega_t^2}{4\Delta_t} = 0 \tag{4.38}$$

holds and use the anti-commutation property $\sigma_i^+\sigma_i^- = \mathbf{1}_i - \sigma_i^-\sigma_i^+$, then

$$\frac{\Omega_r^2}{4\Delta_r}\sigma_i^+\sigma_i^- + \frac{\Omega_s^2}{4\Delta_s}\sigma_i^-\sigma_i^+ + \frac{\Omega_t^2}{4\Delta_t}\sigma_-^+\sigma_+^-) = \frac{\Omega_r^2}{4\Delta_r}\mathbf{1}_i \tag{4.39}$$

becomes a constant that can be neglected. This gives us the resulting master equation:

$$\frac{d\rho}{dt} = -i[H_{\text{cav}}, \rho] + \mathcal{L}_{\text{cav}}\rho \tag{4.40}$$

with Hamiltonian

$$H_{cavity} = H_1(a) + H_2(a^\dagger) + H_n(a^\dagger a) \tag{4.41}$$

$$H_1(a) = \sum_{i=1}^{n} \left(\beta_r \sigma_i^+ + \beta_s e^{-i\phi}\sigma_i^-\right) a = \hat{R}^\dagger a \tag{4.42}$$

$$H_2(a^\dagger) = \sum_{i=1}^{n} \left(\beta_r \sigma_i^- + \beta_s e^{i\phi}\sigma_i^+\right) a^\dagger = \hat{R} a^\dagger \tag{4.43}$$

$$H_{\hat{n}}(a^\dagger a) = \sum_{i=1}^{n} \left(\eta_r \sigma^-\sigma^+ + \eta_s \sigma^+\sigma^-\right) a^\dagger a = \hat{Q} a^\dagger a. \tag{4.44}$$

With this complete formulation of the single cavity system, we are now in a position to treat cavities as modular devices that can be linked and coupled with each other. Note that we have chosen to break up the Hamiltonian in terms of cavity operators. The reason is purely mathematical, and will help us eliminate cavity modes in the next section.

4.3 Cascading the Cavities

The second step towards the characterization of the cascaded cavity system is to couple two separate cavities in a unidirectional configuration. Clearly, we can write the master equation of the system in the form

$$\dot{\rho} = -i\left[H_{sys}, \rho\right] + \mathcal{L}_{c_1} + \mathcal{L}_{c_2} + \mathcal{L}_{coupling} \tag{4.45}$$

where $H_{\text{sys}} = H_{\text{cav}_1} + H_{\text{cav}_2}$ is the combined Hamiltonian for the two independent cavities, and $\mathcal{L}_{c_i}$ describes the decay of the i^{th} cavity. In addition, we introduce the extra superoperator $\mathcal{L}_{\text{coupling}}$ to describe the unidirectional coupling between the two cavities:

$$\mathcal{L}_{\text{coupling}}\rho = -2\sqrt{\epsilon\kappa_1\kappa_2}\left(\left[a_2^\dagger, a_1\rho\right] + \left[\rho a_1^\dagger, a_2\right]\right) \tag{4.46}$$

Here, κ_1 and κ_2 denote the cavity decay rates of the first and second cavities respectively. We have also introduced the parameter $0 \leq \epsilon \leq 1$ to represent coupling efficiency, with perfect coupling when $\epsilon = 1$.

The derivation of $\mathcal{L}_{\text{coupling}}$ is non-trivial, and a detailed description of the process would take us too far afield. Both Carmichael [29]and Gardiner [30] give excellent accounts of the process. For the sake of clarity, we also state the Hamiltonian H_{sys} explicitly:

$$H_{\text{sys}} = \sum_{k=1,2}\left(H_{a_k}(a_k) + H_{a_k^\dagger}(a_k^\dagger) + H_{\hat{n}_k}(\hat{n}_k)\right) \tag{4.47}$$

$$H_{a_k}(a_k) = \sum_{i=1}^{n}\left(\beta_{r,k}\sigma_{i,k}^+ + \beta_{s,k}e^{-i\phi}\sigma_{i,k}^-\right)a = \hat{R}_k^\dagger a_k \tag{4.48}$$

$$H_{a_k^\dagger}(a_k^\dagger) = \sum_{i=1}^{n}\left(\beta_{r,k}\sigma_{i,k}^- + \beta_{s,k}e^{i\phi}\sigma_{i,k}^+\right)a^\dagger = \hat{R}_k a_k^\dagger \tag{4.49}$$

$$H_{\hat{n}_k}(\hat{n}_k) = \sum_{i=1}^{n}\left(\eta_{r,k}\sigma_{i,k}^-\sigma_{i,k}^+ + \eta_{s,k}\sigma_k^+\sigma_k^-\right)a_k^\dagger a_k = \hat{Q}_k a_k^\dagger a_k. \tag{4.50}$$

4.3.1 Adiabatic Elimination of Cavity Modes

The final step of the problem is to trace out the cavity modes by invoking the fact that light escapes the cavities quickly with respect to the interaction time with the atoms. In this regime, we can assume that

$$\rho = \rho_c^{ss} \otimes \rho_a. \tag{4.51}$$

That is, the evolution timescale of the cavity mode is much faster than that of the atoms, and thus the state of the cavities is always a steady state.

Applying the technique given by Parkins and Kimble [31], we can isolate the density operater of the atom via the equation

$$\dot{\rho_a} = \text{Tr}_c\left\{\mathcal{L}_0\int_0^\infty d\tau e^{\mathcal{L}_c\tau}\mathcal{L}_0\rho_c^{ss}\right\}\rho_a. \tag{4.52}$$

Here $\dot{\rho_a}$ denotes the density operator of the atomic system only, and $\mathcal{L}_c = \mathcal{L}_{c_1} + \mathcal{L}_{c_2} + \mathcal{L}_{\text{coupling}}$. A derivation of the equation is available in Gardiner [32].

Substituting (4.47) into the above, and expanding out the commutator, and applying the Bad Cavity condition, we can derive a dynamic equation for ρ_m in the form:

$$
\begin{aligned}
\dot{\rho_m} = -\Bigg(& \int_0^\infty d\tau \mathrm{Tr}\left[a_1 e^{\mathcal{L}_c\tau}(a_1^\dagger \rho_c^{ss})\right](\hat{R}_1^\dagger \hat{R}_1 \rho_m - \hat{R}_1 \rho_m \hat{R}_1^\dagger) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_1 e^{\mathcal{L}_c\tau}(\rho_c^{ss} a_1^\dagger)\right](\rho_m \hat{R}_1 \hat{R}_1^\dagger - \hat{R}_1^\dagger \rho_m \hat{R}_1) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_1^\dagger e^{\mathcal{L}_c\tau}(a_1 \rho_c^{ss})\right](\hat{R}_1 \hat{R}_1^\dagger \rho_m - \hat{R}_1^\dagger \rho_m \hat{R}_1) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_1^\dagger e^{\mathcal{L}_c\tau}(\rho_c^{ss} a_1)\right](\rho_m \hat{R}_1^\dagger \hat{R}_1 - \hat{R}_1 \rho_m \hat{R}_1^\dagger) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_2 e^{\mathcal{L}_c\tau}(a_2^\dagger \rho_c^{ss})\right](\hat{R}_2^\dagger \hat{R}_2 \rho_m - \hat{R}_2 \rho_m \hat{R}_2^\dagger) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_2 e^{\mathcal{L}_c\tau}(\rho_c^{ss} a_2^\dagger)\right](\rho_m \hat{R}_2 \hat{R}_2^\dagger - \hat{R}_2^\dagger \rho_m \hat{R}_2) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_2^\dagger e^{\mathcal{L}_c\tau}(a_2 \rho_c^{ss})\right](\hat{R}_2 \hat{R}_2^\dagger \rho_m - \hat{R}_2^\dagger \rho_m \hat{R}_2) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_2^\dagger e^{\mathcal{L}_c\tau}(\rho_c^{ss} a_2)\right](\rho_m \hat{R}_2^\dagger \hat{R}_2 - \hat{R}_2 \rho_m \hat{R}_2^\dagger) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_1 e^{\mathcal{L}_c\tau}(a_2^\dagger \rho_c^{ss})\right](\hat{R}_1^\dagger \hat{R}_2 \rho_m - \hat{R}_2 \rho_m \hat{R}_1^\dagger) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_1 e^{\mathcal{L}_c\tau}(\rho_c^{ss} a_2^\dagger)\right](\rho_m \hat{R}_2 \hat{R}_1^\dagger - \hat{R}_1^\dagger \rho_m \hat{R}_2) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_2^\dagger e^{\mathcal{L}_c\tau}(a_1 \rho_c^{ss})\right](\hat{R}_2 \hat{R}_1^\dagger \rho_m - \hat{R}_1^\dagger \rho_m \hat{R}_2) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_2^\dagger e^{\mathcal{L}_c\tau}(\rho_c^{ss} a_1)\right](\rho_m \hat{R}_1^\dagger \hat{R}_2 - \hat{R}_2 \rho_m \hat{R}_1^\dagger) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_1^\dagger e^{\mathcal{L}_c\tau}(a_2 \rho_c^{ss})\right](\hat{R}_1 \hat{R}_2^\dagger \rho_m - \hat{R}_2^\dagger \rho_m \hat{R}_1) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_1^\dagger e^{\mathcal{L}_c\tau}(\rho_c^{ss} a_2)\right](\rho_m \hat{R}_2^\dagger)\hat{R}_1 - \hat{R}_1 \rho_m \hat{R}_2^\dagger) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_2 e^{\mathcal{L}_c\tau}(a_1^\dagger \rho_c^{ss})\right](\hat{R}_2^\dagger \hat{R}_1 \rho_m - \hat{R}_1 \rho_m \hat{R}_2^\dagger) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_2 e^{\mathcal{L}_c\tau}(\rho_c^{ss} a_1^\dagger)\right](\rho_m \hat{R}_1 \hat{R}_2^\dagger - \hat{R}_2^\dagger \rho_m \hat{R}_1) + \ldots \Bigg).
\end{aligned}
\tag{4.53}
$$

Here the dots signify all other possible combinations of the operators $\hat{R}_k a_k^\dagger$, $\hat{R}_k^\dagger a_k^\dagger$ and $\hat{Q}_k a_k^\dagger a_k$. That is, the formal expansion of this equation actually contains $4 \times 6^2 = 144$ terms.

While simplifying such an equation may look daunting, nearly all of the terms do vanish, and

the rest can be calculated via the Quantum Regression Formula presented by Carmichael [12].

$$\langle X(t)Y(t+\tau)\rangle_{ss} = \lim_{t\to\infty}\langle X(t)Y(t+\tau)\rangle = \mathrm{Tr}(Ye^{\mathcal{L}_c\tau}(\rho X)) \tag{4.54}$$

$$\langle X(t+\tau)Y(t)\rangle_{ss} = \lim_{t\to\infty}\langle X(t+\tau)Y(t)\rangle = \mathrm{Tr}(Xe^{\mathcal{L}_c\tau}(Y\rho)) \tag{4.55}$$

Thus, the evaluation of our reduced master equation is reduced to the evaluation of all possible second order correlations $\langle O_1(t+\tau)O_2(t)\rangle_{ss}$. The process is conceptually easy, but technically laborious. We state the non-zero correlations below:[1]

$$\left\langle a_1(t)a_1^\dagger(t+\tau)\right\rangle_{ss} = = e^{-\kappa t}$$

$$\left\langle a_1(t)a_2^\dagger(t+\tau)\right\rangle_{ss} = -2\kappa\tau e^{-\kappa\tau} \tag{4.56}$$

$$\left\langle a_2(t+\tau)a_1^\dagger(t)\right\rangle_{ss} = = -2\kappa\tau e^{-\kappa\tau}. \tag{4.57}$$

It is worth noting that correlations between the second and first cavity are time asymmetric, which corresponds to our initial condition of uni-directional coupling. Applying the results of the Quantum Regression Formula, we obtain the required traces

$$\int_0^\infty d\tau \mathrm{Tr}\left[a_1e^{\mathcal{L}_c\tau}(a_1^\dagger\rho_c^{ss})\right] = \int_0^\infty d\tau \mathrm{Tr}\left[a_2e^{\mathcal{L}_c\tau}(a_2^\dagger\rho_c^{ss})\right] = \int_0^\infty e^{-\kappa\tau}d\tau = \frac{1}{\kappa},$$

$$\int_0^\infty d\tau \mathrm{Tr}\left[a_2e^{\mathcal{L}_c\tau}(a_1^\dagger\rho_c^{ss})\right] = \int_0^\infty d\tau \mathrm{Tr}\left[a_2e^{\mathcal{L}_c\tau}(a_1^\dagger\rho_c^{ss})\right] = \int_0^\infty -2\kappa e^{-\kappa\tau}d\tau = -\frac{2}{\kappa}.$$

We substitute into the formal form of the master equation to obtain explicit dynamics for the density operator of the atomic states

$$\dot\rho = \frac{1}{\kappa}\sum_{i=1,2}\left(2R_i\rho R_i^\dagger - R_i^\dagger R_i\rho - \rho R_i^\dagger R_i\right) \tag{4.58}$$

$$+\frac{2\sqrt{\epsilon}}{\kappa}(\rho R_1^\dagger R_2 - R_2\rho R_1^\dagger + R_2^\dagger R_1\rho - R_1\rho R_2^\dagger). \tag{4.59}$$

4.4 Interpretation of the Cascaded Cavity Equation

The first step to understanding such an equation is to understand its components, to gain a intuitive grasp of what dynamics the equation represents. Firstly, let us consider a physical interpretation of the operator

$$R_k^\dagger = \beta_{r,k}\left(\sum_{i=1}^n \sigma_{i,k}^+\right) + \beta_{s,k}e^{-i\phi}\left(\sum_{i=1}^n \sigma_{i,k}^-\right). \tag{4.60}$$

The operator is completely symmetric with respect to σ_i, and hence any dynamics caused by R will be symmetric to all intra-cavity atom interchanges. Now as the dynamics of the

[1] Detailed calculations can be observed in Appendix A for readers who are (very) technically inclined.

system is completely characterized by this operator, the symmetry cannot be violated by time evolution. In effect, atoms within the cavity remain indistinguishable for all time.

For example, let us consider the case where all atoms within the cavity were first prepared in the state $|0\rangle$, an application of R will give, after normalisation, the state:

$$\frac{1}{\sqrt{N}}\left(|1000\ldots0\rangle + |0100\ldots0\rangle + \ldots + |000\ldots1\rangle\right), \tag{4.61}$$

which is simply a superposition of all states with 1 atom in $|1\rangle$.

Each cavity is in fact algebraically homeomorphic to a physical system with n spin-1/2 particles, where we have the direct mapping

$$\begin{aligned} |1\rangle &\leftrightarrow |\uparrow\rangle \\ |0\rangle &\leftrightarrow |\downarrow\rangle \\ \sum_{i=1}^{n}\sigma_{i,k}^{-} &\leftrightarrow S_k \\ \sum_{i=i}^{n}\sigma_{i,k}^{+} &\leftrightarrow S_k^{\dagger}, \end{aligned} \tag{4.62}$$

where S and $S^{\dagger}$ are the ladder lowering and raising operators for a standard quantum mechanical system with total angular momentum of $\frac{n}{2}$.

Therefore, the conservation of symmetry equates to the conservation of total angular momentum, and the operators S_k and $S_k^{\dagger}$ are simply angular momentum ladder operators obeying all the standard commutation relations.

We can now rewrite R_k as

$$R_k = \beta_{r,k}^{*}S + \beta_{s,k}^{*}e^{i\phi}S^{\dagger}, \tag{4.63}$$

that is, a superposition of the actions of either flipping one atom from $|1\rangle$ into $|0\rangle$ or vice versa. Therefore the parameters $\beta_{r,k}$ and $\beta_{s,k}$ can be interpreted as the effective rates of transition between $|0\rangle$ and $|1\rangle$ for the k^{th} cavity.

Regardless of which transition occurs, a photon is created within the cavity mode, and is then emitted from the cavity by the bad cavity approximation. Hence, we finally obtain a physical interpretation of R_k as the action of a photon emission from the k^{th} cavity.

Finally, in order to simplify notation, we will absorb the cavity decay rate κ into our transition rate constants via the transformation

$$\frac{\beta_{s,k}}{\sqrt{\kappa}}e^{-i\phi} \rightarrow \beta_{s,k}, \tag{4.64}$$

so that the master equation

$$\dot{\rho} = \sum_{i=1,2} \left(2R_i \rho R_i^\dagger - R_i^\dagger R_i \rho - \rho R_i^\dagger R_i \right) \tag{4.65}$$

$$+2\sqrt{\epsilon}(\rho R_1^\dagger R_2 - R_2 \rho R_1^\dagger + R_2^\dagger R_1 \rho - R_1 \rho R_2^\dagger). \tag{4.66}$$

can be written more succinctly, with

$$R_k^\dagger = \beta_{r,k} \left(\sum_{i=1}^{n} \sigma_{i,k}^+ \right) + \beta_{s,k} \left(\sum_{i=1}^{n} \sigma_{i,k}^- \right). \tag{4.67}$$

This master equation completely characterizes the dynamics of the computational (ground) states for the system. That is, we have successfully eliminated both the excited states and cavity modes using adiabatic considerations. We are now in a position to analyze the dynamics of the cascaded cavity system.

Chapter 5

Quantum Cycles

> "If a tree falls in the forest and no one is around to hear it, does it make a sound?"
> -*Unknown*

5.1 Introduction

Now that we have a theory to investigate the time evolution of entanglement in general open systems, and the details of a particular open system of interest, we are in a position to piece together the puzzle.

In this chapter, we consider the Cascaded Qubit System, the simplest case of the Cascaded Cavity System, where we have only one atom in each individual cavity. While the entanglement behavior at steady state of such a system is already well known, its entanglement during time evolution has not yet been characterized exactly.

In this section, we apply the method of quantum trajectories to the Cascaded Cavity System, to show how entanglement can be quantized in a measure that is transparent to direct physical interpretation. The very act of measurement leads us to a new phenomenon where the system undergoes stochastic cycles between fully entangled states. This phenomenon, we term as Quantum Cycles.

5.2 The Steady State Model

The Cascaded Qubit System is not a new system, and has already been proposed as a promising method to generate arbitrary states in the entanglement-entropy plane [27]. In particular, the system is capable of generating pure entangled states in the limit where coupling loss was negligible. In this section, we will outline these results, and show how they can be extended.

The system is a special case of the Cascaded Cavity System, where we set $n = 1$, to obtain the master equation

$$\begin{aligned}\dot{\rho} = &\sum_{i=1,2}\left(2R_i\rho R_i^\dagger - R_i^\dagger R_i\rho - \rho R_i^\dagger R_i\right)\\ &+ 2\sqrt{\epsilon}(\rho R_1^\dagger R_2 - R_2\rho R_1^\dagger + R_2^\dagger R_1\rho - R_1\rho R_2^\dagger),\end{aligned} \tag{5.1}$$

where the cavity operator

$$R_k = \beta_{r,k}^* S_k + \beta_{s,k}^* S_k^\dagger \tag{5.2}$$

is now just a sum of Pauli matrices.

5.2.1 Entanglement at Steady State

The first step with any dynamical system is to consider the long term behaviour. In this case, we can envision that for large t, the input and output to the cascaded system will balance, forming a steady state. The state can be easily evaluated by converting all superoperators into matrices[1] and solving the linear equation

$$\mathcal{L}\rho = 0. \tag{5.3}$$

For simplicity, we will limit ourselves to the case where the parameters of the two atom-cavity systems are identical, i.e: $\beta_r = \beta_{r,1} = \beta_{r,2}$, $\beta_s = \beta_{s,1} = \beta_{s,2}$. This special case is rather reasonable, given that the cavities themselves can considered as modules that make up a complex system.[2] With this restriction, the steady state for this system can be easily determined [27].

$$\rho_{ss} = \begin{bmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{bmatrix} \tag{5.4}$$

Where

$$\begin{aligned}
\rho_{11} &= \left[|\beta_r|^6 + (1+\epsilon-4\epsilon^2)|\beta_r|^2|\beta_s|^4 + \epsilon|\beta_r|^4|\beta_s|^2)\right]/D,\\
\rho_{22} &= |\beta_r|^2|\beta_s|^2(1-\epsilon)\left[|\beta_r|^2 + (1+4\epsilon)|\beta_s|^2\right]/D,\\
\rho_{33} &= |\beta_r|^2|\beta_s|^2(1-\epsilon)\left[|\beta_s|^2 + (1+4\epsilon)|\beta_r|^2\right]/D,\\
\rho_{44} &= \left[|\beta_s|^6 + (1+\epsilon-4\epsilon^2)|\beta_s|^2|\beta_r|^4 + \epsilon|\beta_s|^4|\beta_r|^2)\right]/D,\\
\rho_{14} &= \sqrt{\epsilon}\beta_r\beta_s^*\left[|\beta_r|^4 + (2-4\epsilon)|\beta_r|^2|\beta_s|^4 + |\beta_s|^4\right]/D,\\
\rho_{23} &= 2\sqrt{\epsilon}(1-\epsilon)|\beta_r|^2|\beta_s|^2(|\beta_r|^2 + |\beta_s|^2)/D.
\end{aligned}$$

where D is a normalization constant chosen so that the trace of the matrix is 1. Consider the steady state where the inter-cavity coupling is perfect ($\epsilon = 1$). In such an instance, the

[1] The exact implementation of this can be found in the documentation for Sze Tan's Quantum Optics Toolbox [33].

[2] We will also show later that this if one of the conditions that allow the existence of pure steady states.

second cavity no longer emits photons, and as there is no loss to the environment, the system is effectively closed. Thus, the steady state turns out to be a pure state, and can be easily evaluated:

$$|\phi_{ss}\rangle = \frac{1}{\sqrt{|\beta_r|^2 + |\beta_s|^2}} \left(\beta_r^* |00\rangle + \beta_s^* |11\rangle\right). \tag{5.5}$$

Recall that $|\beta_r|$ and $|\beta_s|$ are the effective transition rates $|1\rangle \to |0\rangle$ and vice versa, and hence this solution makes sense. That is, $\beta_s = 0$ would signify no transitions from $|0\rangle \to |1\rangle$, and hence would would expect the steady state to be $|00\rangle$.

In order to generate maximal entanglement, we want the steady state to be a Bell state. Clearly, this occurs when $\beta_r = \beta_s$. In fact, this condition yields a lot of interesting behavior, and hence we define it formally for convenience.

Definition 10 (Resonance) *A Cascaded Cavity System is resonant if and only if $\beta_r = \beta_s$, where β_r and β_s are effective qubit transition rates defined above.*

The above results imply at first glance that to achieve optimal entanglement, we should set our system parameters as closely as we can to the resonance condition.

5.2.2 Uniqueness and Rates of Convergence

Prior to a painful mathematical analysis of a system, one should never forget to step back and look at things from a physical perspective. Not only does it prevent careless mistakes, but it also gives insight into why the mathematics says what it says.

Intuitively, we may expect interesting behaviour at resonance. For in many physical systems, the equality of transitions rates often bring up degenerate behaviour. To check this mathematically, we turn to the field of eigenvalue analysis. Consider an arbitrary system of linear differential equations

$$\dot{\mathbf{y}} = A\mathbf{y}, \tag{5.6}$$

where A is a matrix and $\mathbf{y}$ a vector in $\mathbb{R}^n$. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the normalised eigenvectors of A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the general solution for the system can be written in the form [35]:

$$\mathbf{y}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \mathbf{v}_i. \tag{5.7}$$

Without loss of generality, assume that λ_i are written in decreasing order. Since $y(t)$ is already constrained by the normalization condition, the dynamic equation above must allow at least one degree of freedom and steady state, and hence features at least one zero eigenvalue.

Let us assume the steady state is attractive. Then $\mathrm{Re}(\lambda_i) \leq 0$ for all $i \neq 1$, and

$$\mathbf{y}(t) \quad = \quad \mathbf{y}_{ss} + \sum_{i=2}^{n} c_i e^{\lambda_i t} \mathbf{v}_i. \tag{5.8}$$

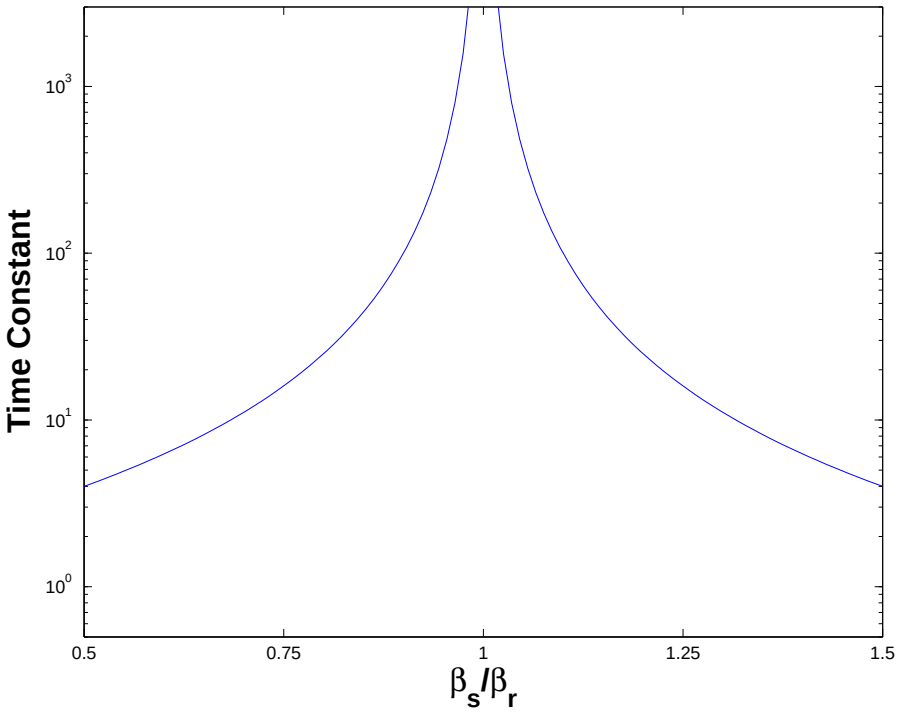

Figure 5.1: The timescale on which the cascaded qubit system settles to steady state for different ratios of β_s/β_r with $\beta_r = 1$. Clearly, the closer the system is to resonance, the longer the process will take.

In the limit where $t \to \infty$, we have

$$\mathbf{y}(t) - \mathbf{y}_{ss} \approx c_2 e^{\lambda_2 t} \mathbf{v}_2. \tag{5.9}$$

Therefore, we can define the time constant $\tau = 1/\lambda_2$ as a good indication of the timescale on which the system evolves towards steady state.[3]In particular, for $t \geq 3\tau$, we can say that the system is approximately at steady state. It is a simple matter to determine the eigenvalues and hence time constant numerically. Figure 5.1 plots the time constant for different values of β_s/β_r, with normalisation $\beta_r = 1$.

It is clear to see that the rate at which the system converges to steady state is heavily dependent on β_s/β_r. In particular, the system never reaches steady state at the exact point of resonance. Unfortunately, this is also the exact condition that will produce maximal entanglement.

Steady state analysis demonstrates that there is a distinct tradeoff that we will have to consider. The longer we let the system evolve, the greater the amount of energy we need to

[3]Of course, we have simplified matters considerable by disregarding the case where λ_2 and λ_3 are complex conjugates. Fortunately, in such instances, λ_2 can still be used as an indication of the timescale of evolution since numerics show that the graph of the time constant is smooth. i.e if λ_2 and λ_3 were complex conjugates, they would be remain complex conjugates for all values of β_s/β_r plotted.

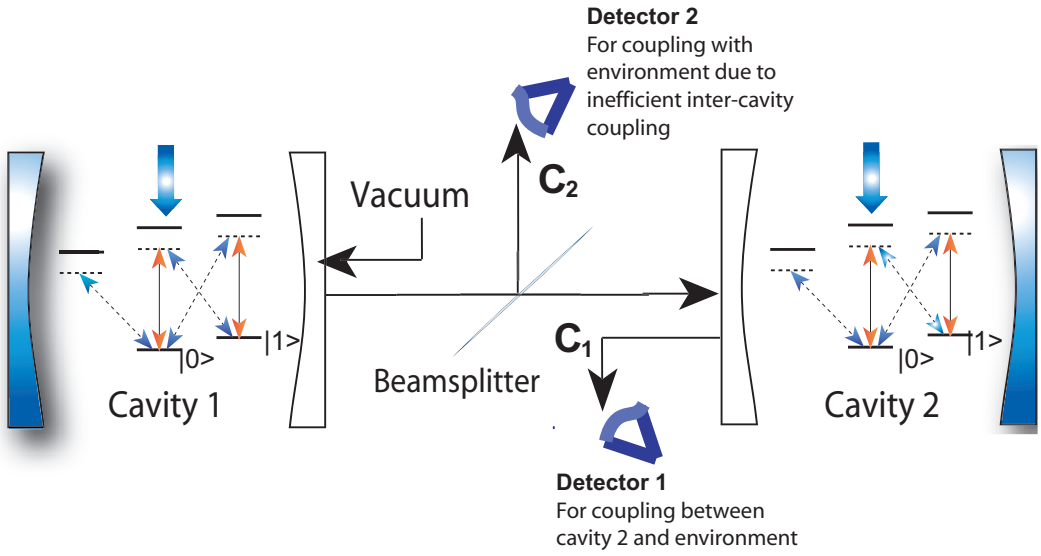

Figure 5.2: Conceptual photon detectors for unravelling the master equation. Faraday isolators have been omitted for the sake of clarity.

supply to the system. To generate a Bell state exactly, we will need to, in principle, run the system for a infinite amount of time. The solution is to compute the level of entanglement that can be reached after a certain fixed amount of time, and work out optimal values for this trade-off. However, this has already been investigated [27].

Alternatively, one could consider a radically different approach. At each time interval, the system may emits photons and therefore inevitably interacts with the environment. That is, the system and environment becomes superpositions of states where the system either emitted, or did not emit a photon. Discarding the information about the state of environment gives rise to information loss. Therefore, the cascaded cavity system is, in general, described by a mixed state.

Detection of these photons would induce a pure state unravelling onto the system, allowing us to characterize its entanglement completely. We can then ask the question, how was the system entangled throughout its evolution?

5.3 Trajectory Theory of the Cascaded System

As with any open system, the first step in unravelling the master equation is to identify the points of coupling with the environment. The first is obvious, the output from the second cavity. To measure this interaction, let us consider the existence an ideal photon detector at the output, which we call *Detector 1*.

The second point of coupling is more subtle. Notice that our model does not assume that the inter-cavity coupling is perfect, that is, only a fraction $\sqrt{\epsilon}$ of the output field from the

first cavity ever makes it into the second cavity. Physically, this loss is caused by non-ideal transmissivity of the Faraday isolators and cavity mirrors. All of these cause incident photons to be scattered into the environment in some random, uncontrollable fashion. We will assume the existence of a second photon detector that collects all of the scattered light, which we call *Detector 2*. Formally, this is equivalent to the assumption that while all apparatus is ideal, there exists a conceptual beamsplitter between the cavities. The configuration is demonstrated in Figure 5.2.

We are now in a position to examine the trajectory formalism for the cascaded qubit system, such that the evolution of the system is reduced to a pure state that is dependent on the detection histories of the two detectors.

5.3.1 The Trajectory Formalism

A careful observation of the master equation reveals that it can be written in the general standard quantum trajectory formalism:

$$\begin{aligned} \dot{\rho} &= (\mathcal{L}_0 + S)\rho, \\ \mathcal{L}_0\rho &= -i\,[H,\rho] - (C_1^\dagger C_1\rho + \rho C_1^\dagger C_1) - (C_2^\dagger C_2\rho + \rho C_2^\dagger C_2), \\ S\rho &= C_1\rho C_1^\dagger + C_2\rho C_2^\dagger \end{aligned} \tag{5.10}$$

with operators operators

$$C_1 = \sqrt{2}(R_1 - \sqrt{\epsilon}R_2), \qquad C_2 = \sqrt{2(1-\epsilon)}R_2, \qquad H = i\sqrt{\epsilon}(R_2^\dagger R_1 - R_1^\dagger R_2).W \tag{5.11}$$

As one intuitively expects, we have two collapse operators, representing each of the two detectors that we have introduced into our model. In the limit of perfect coupling, that is, when $\epsilon = 1$, $C_2 = 0$. Hence $C_2$is early associatedwith the second detector, and C_1 the first.

All the results from trajectory theory now carry through. The evolution of the system can be described in terms of a pure state under the effective non-hermitian Hamiltonian

$$H_{eff} = H - \frac{i}{2}\left(C_1^\dagger C_1 + C_2^\dagger C_2,\right) \tag{5.12}$$

coupled with the probabilistic jumps

$$|\phi_c\rangle \;\rightarrow\; C_1|\phi_c\rangle \qquad \text{and} \qquad |\phi_c\rangle \rightarrow C_2|\phi_c\rangle \tag{5.13}$$

that occur during interval $(t, t+dt)$ with probabilities

$$p_1(t)dt = \frac{\langle\phi|C_1^\dagger C_1|\phi\rangle}{\langle\phi||\phi\rangle}dt \qquad \text{and} \qquad p_2(t)dt = \frac{\langle\phi_c|C_2^\dagger C_2|\phi_c\rangle}{\langle\phi_c||\phi_c\rangle}dt \tag{5.14}$$

respectively.

Thus, in the trajectory formalism, we consider an experiment where two such ideal detectors are employed, and each photon detected and recorded. Given such a history of detector 'clicks', one would always have complete information about the state of the system, and hence be able to characterize its entanglement in a fundamental and physical fashion. Fortunately, such an experiment need not be done in real life, as numerical methods work particular well in the trajectory formalism.

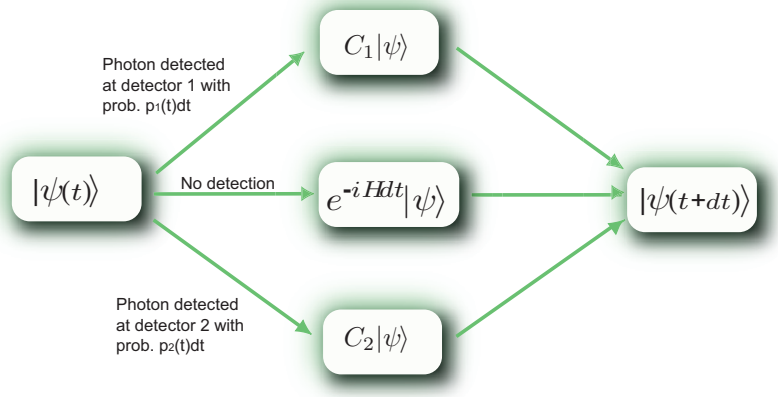

Figure 5.3: Flowchart of the stochastic evolution of a quantum trajectory during one time step.

5.3.2 Numerical Method for Quantum Trajectories

Consider the simulation of a standard master equation. For a system in a Hilbert space of dimension d, the density operator is of dimension d^2. In contrast, a pure state in such a system is only of dimension d. That is, the computation time of simulating a master equation grows quadratically faster than that of a single pure state trajectory, which makes trajectories an efficient and attractive approach to numerical simulations.

We can easily implement the evolution of the system over the interval of $t = [t_0, t_f]$ by considering the evolution of the system over n discrete timesteps of size $\Delta t = (t_f - t_0)/n$. Provided Δt satisfies

$$p_1(t)\Delta t \ll 1 \qquad \text{and} \qquad p_2(t)\Delta t \ll 1. \tag{5.15}$$

At the beginning of each time step, the algorithm rolls random numbers to see whether a collapse occurs, and if so, which of the two collapses has occurs. If there are no collapses, it evolves the wavefunction through free evolution under H_{eff}, otherwise it applies the appropriate jump operator to the wavefunction. A summary of the procedure is shown in Figure 5.3. We will omit the technical details of the code, for that is specific to the language, and a very robust implementation is already available in the Quantum Optics Toolbox designed by Sze Tan [33].

In practice, there are two general aims for running a trajectory simulation. The first is to simulate a master equation approximately by taking a large number of trajectories. This has the advantage that for practical purposes, one does not need perfect accuracy, and stimulating a sizeable number of trajectories is still computationally much faster than numerical integration of a master equation.

In addition the trajectory approach produces an unravelling with a distinct interpretation. Consider, for example, taking the average of the entanglement measured for each individual

pure trajectory, this is the expected entanglement you would get at time t if you were the repeat the experiment many times. In contrast, one cannot take an exact measurement of entanglement for a mixed state, and hence cannot use the master equation to give a direction measure of entanglement with a clear experimental interpretation.

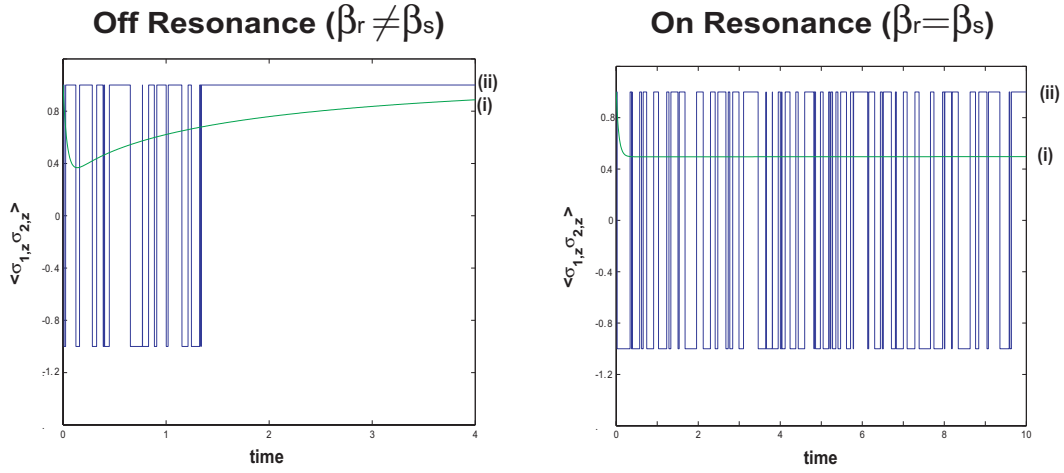

Figure 5.4: The dynamics of the ensemble average (i) and a single quantum trajectory (ii) for $\epsilon = 1$.

However, in many situations, we actually gain much more insight about how a system behaves by studying the evolution of a single trajectory. An ensemble average may only show us how a system evolves on average, but it does not shed light on how this average arises, nor does it give us insight into the expected results of individual experiments. We will see that this is especially true for the cascaded qubit system, where the properties of single trajectories can give us complete understanding about its transient properties.

5.3.3 A Glimpse of Cycles

Let us first restrict ourselves to the case where the coupling is ideal. Therefore, the dynamical system simplifies to just one jump operator. The numerical techniques mentioned above can be easily implemented to run simulations on a single trajectory. However, displaying these trajectories in a visually informative matter is slightly more challenging.

A plot of all components of the quantum state is unnecessarily messy. Instead let us consider the evolution of the expectation value of the operator $\sigma_{1,z}\sigma_{2,z}$, where $\sigma_{i,z}$ is the Pauli operator such that

$$\sigma_{i,z}|1\rangle = |1\rangle, \qquad \sigma_{i,z}|0\rangle = -|0\rangle. \tag{5.16}$$

This expectation value has a number of nice properties. For example

$$\langle \sigma_{1,z}\sigma_{2,z} \rangle_{ss} = 1 \tag{5.17}$$

at steady state, regardless of the values of β_r and β_s, which makes it easier to compare rates of convergence for different system parameters.

Figure 5.4 contrasts the differences between the dynamics of the master equation and that of a single trajectory. The master equation is just a differential equation, and thus the evolution is completely smooth and tends asymptotically towards the steady state. On the other hand, the quantum trajectory undergoes a series of collapses between the two extreme values of $\langle \sigma_{1,z}\sigma_{2,z} \rangle_{ss}$, occurring at each photon detection.

Provided the system is not at resonance, photon emissions eventually stop and the trajectory settles on the steady state, whereas the photon emissions never cease at resonance. Physically, this looks plausible, as it simply implies that the atoms within our cavity continually switch between states $|0\rangle$ and $|1\rangle$, emitting a photon out of the system with each transition. This cyclic behaviour would be completely invisible had we considered an ensemble average, a vivid demonstration of how single trajectories often give us great insight into a system.

5.4 Qualitative Quantum Cycles

The oscillatory behaviour featured in Figure (5.4) hints at a simple cyclic process. So simple, in fact, that we can understand why it occurs without resorting to numerics. In this section, we explain why these cycles occur from a qualitative perspective.

5.4.1 The Cascaded Phase Space

Qualitative analysis of dynamical systems is not a new subject, and phase portraits have been used for decades to understand classical dynamical systems. Our quantum system, however, requires something slightly more elaborate. Firstly, it features the stochastic element of quantum collapses, and secondly, it resides in $\mathbb{R}^4$. Due to limitations of the human mind, standard methods of visualization may serve only to cause headaches.

Fortunately, the numerical results in Figure (5.4) give us a clue. In particular, the the value of $\langle \sigma_{1,z}\sigma_{2,z} \rangle$ is conserved during periods of free evolution. Therefore, we can define the positive and negatively correlated subspaces

$$E^{\pm} = \{|\phi\rangle : \langle \sigma_{1,z}\sigma_{2,z} \rangle = \pm 1\}\,. \tag{5.18}$$

If the system is ever located within either subspace, the free evolution of the system is constrained to that subspace. Quantum collapses can now be represented as jumps between E^+ and E^-. Noting that

$$E^+ = \text{span}\{|00\rangle, |11\rangle\}, \quad E^- = \text{span}\{|10\rangle, |01\rangle\} \tag{5.19}$$

are both 2-dimensional, we have managed to break up a 4-dimensional space into two 2-dimensional planes where phase portraits can be easily plotted; linked together by quantum collapses. We define this representation as the cascaded phase space.

5.4.2 Dynamics of Free Evolution

As free evolution is constrained in the positive and negatively correlated subspaces, we can use standard phase portrait techniques on E^+ and E^- to characterize the behaviour of the system.

For the sake of algebraic simplicity, let us define $r = \beta_s/\beta_r$ and set $\beta_r = 1$. This can be done without any loss of generality, as it is equivalent to a substitution $\beta_r t \to t$, introducing coordinates of normalized time. The master equation is reduced to

$$\begin{aligned}\dot{\rho} = &\sum_{i=1,2}\left(2R_i\rho R_i^\dagger - R_i^\dagger R_i\rho - \rho R_i^\dagger R_i\right)\\ &+ 2\sqrt{\epsilon}(\rho R_1^\dagger R_2 - R_2\rho R_1^\dagger + R_2^\dagger R_1\rho - R_1\rho R_2^\dagger),\end{aligned} \tag{5.20}$$

with cavity operator

$$R_k^\dagger = S_k + rS_k^\dagger, \tag{5.21}$$

and resonance condition $r = 1$.

At this point, it is useful to convert to matrix notation such that a pure state of the system is a 4 vector $|\phi\rangle$, such that

$$|\phi\rangle = (c_{11}, c_{10}, c_{01}, c_{00})^T = \sum_{i,j=0,1} c_{ij}|ij\rangle \tag{5.22}$$

whereby, the system operators can be written as

$$\begin{aligned}C_1 &= \begin{pmatrix} 0 & -r & r & 0\\ -1 & 0 & 0 & r\\ 1 & 0 & 0 & -r\\ 0 & 1 & -1 & 0\end{pmatrix},\\ C_1^\dagger C_1 &= 2\begin{pmatrix} 2 & 0 & 0 & -2r\\ 0 & 1+r^2 & -(1+r^2) & 0\\ 0 & -(1+r^2) & 1+r^2 & 0\\ -2r & 0 & 0 & 2r^2\end{pmatrix}.\end{aligned}$$

The evolution of $|\phi\rangle$ can now be written as a linear differential equation of four variables

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\phi\rangle = -iH_{eff}|\phi\rangle &= \left[\frac{1}{i}H - \frac{1}{2}C_1^\dagger C_1\right]|\phi\rangle\\ &= \begin{pmatrix} -2 & 0 & 0 & 2r\\ 0 & -(1+r^2) & 2r^2 & 0\\ 0 & 2 & -(1+r^2) & 0\\ 2r & 0 & 0 & -2r^2\end{pmatrix}|\phi\rangle.\end{aligned} \tag{5.23}$$

Recall that we have already shown that free evolution is constrained within E^+ and E^-. This is reflected by the fact that this differential equation can be completely decoupled into the two separate subspaces. Elements which begin in one of the subspaces will remain there for all time under free evolution.

In effect, we can write $|\phi\rangle = |\phi^+\rangle + |\phi^-\rangle$ as a vector sum of two orthogonal components, $|\phi^+\rangle \in E^+$ and $|\phi^-\rangle \in E^-$, and treat the dynamics of the two parts separately. i.e,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\phi^+\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c_{11} \\ c_{00} \end{pmatrix} = \begin{pmatrix} -2 & 2r \\ 2r & -2r^2 \end{pmatrix}\begin{pmatrix} c_{11} \\ c_{00} \end{pmatrix} \tag{5.24}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}|\phi^-\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c_{10} \\ c_{01} \end{pmatrix} = \begin{pmatrix} -(1+r^2) & 2r^2 \\ 2r & -(1+r^2) \end{pmatrix}\begin{pmatrix} c_{10} \\ c_{01} \end{pmatrix}. \tag{5.25}$$

Eigenstates of the dynamic equations above can be easily determined. Physically, these eigenstates correspond to states of the system that are preserved under free evolution. Note that this does not necessarily imply that they are stable states of the system, as they may still undergo quantum collapse.

Recall from Chapter 3 that the chance that a state will not collapse prior to time t is equal to its norm at that time, thus eigenstates with conserved norm are entirely stable. This leads us to a handy definition.

Definition 11 (Classification of Eigenstates) *Let $|\phi\rangle$ be an eigenstate of the effective Hamiltonian H_{eff} with eigenvalue λ. $|\phi\rangle$ is stable iff $\lambda = 0$. Stable eigenstates are steady states of the system. Conversely, if $\lambda < 0$, the eigenstate is unstable, and will eventually undergo quantum collapse.*

We find (unnormalised) eigenvectors and corresponding eigenvalues as follows:

- **Stable eigenstate $|\phi_1\rangle = r|11\rangle + |00\rangle$ with eigenvalue $\lambda_1 = 0$**
  This is the general steady state of the system, in agreement with previous work done by Clark et al. [27].
- **Unstable eigenstate $|\phi_2\rangle = |11\rangle - r|00\rangle$ with eigenvalue $\lambda_2 = -2(1+r^2)$**
  Any states orthogonal to $|\phi_1\rangle$ in E^+ will eventually collapse.
- **(Un)stable $|\phi_3\rangle = |01\rangle + r|10\rangle$ with eigenvalue $\lambda_3 = -(r-1)^2$**
  This state in E^- is unstable unless the system is at resonance, whereby it becomes stable. We will see however, that this state has little effect on the evolution of individual trajectories.
- **Unstable eigenstate $|\phi_4\rangle = -r|10\rangle + |10\rangle$ with eigenvalue $\lambda_4 = -(r+1)^2$**
  A second unstable eigenstate in E^-.

It is worth noting that in the special case of resonance, there are two independent steady states of the system, $|\phi_1\rangle$ and $|\phi_3\rangle$. A cascaded phase portrait is shown in Figure 5.5. This agrees with our previous analysis, where we have already shown that the system does not converge to a unique steady state at resonance.

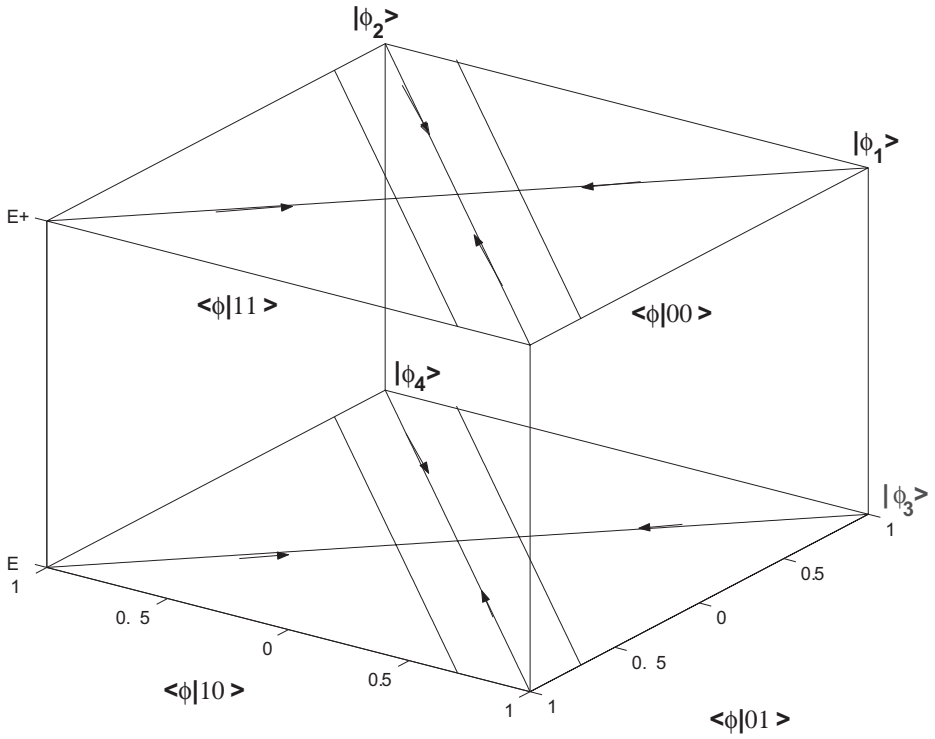

Figure 5.5: The use of cascaded $\mathbb{R}^2$ phase planes is a convenient way to visualize the dynamics of a Cascaded Qubit System. The top and bottom planes represent the phase space of E^+ and E^- respectively. Note that since the state $|\phi\rangle$ is not normalized, the phase space are planes and not lines. In this representation, we plot the phase portrait of free evolution dynamics for the system when $r = 1$. On E^+, $|\phi_1\rangle$ has eigenvalue 0 and is thus stable. In contrast, $|\phi_2\rangle$ has a negative eigenvalue, and thus its norm tends to 0, which implies that it must undergo quantum collapse at some point in time. For an arbitrary initial $|\phi\rangle \in E^+$, free evolution would map the state to $|\phi_1\rangle$ with probability $|\langle\phi|\phi_1\rangle|^2$, but it will also have a finite chance of collapse, given by $|\langle\phi|\phi_2\rangle|^2$. The dynamics on E^- are identical.

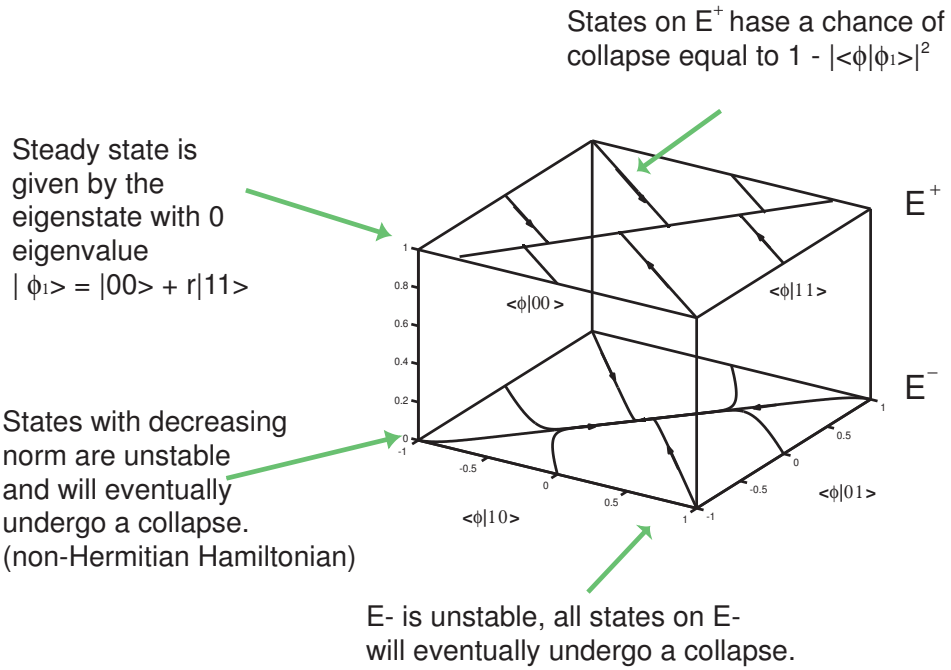

Figure 5.6: Phase space portrait of the cascaded qubit system for the general case where $r \neq 1$.

With two possible steady states, perhaps the first conclusion we might jump to is that the system will settle into a superposition of both, with its exact location dependent on the initial state. However, we will show something quite startling, and unexpected. The stochastic nature of the system can create unique cycles that do not touch either steady state at all.

5.4.3 The Dynamics of Quantum Collapses

Consider an arbitrary state of the cascaded cavity system, which we can write in the form

$$|\phi\rangle = \sum_{i=1}^{4} c_i |\phi_i\rangle, \tag{5.26}$$

where $|\phi_i\rangle$ are the eigenstates of free evolution. Clearly, if the system were to evolve for an infinite amount of time under free evolution, it would be reduced to the form

$$|\phi\rangle = \sum_{j:\lambda_j=0} c_j |\phi_j\rangle \tag{5.27}$$

as all components with negative eigenvalues will vanish. The norm of this wavefunction, $\sum_j |c_j|^2$, is the no-collapse probability. Thus, we arrive at the following proposition

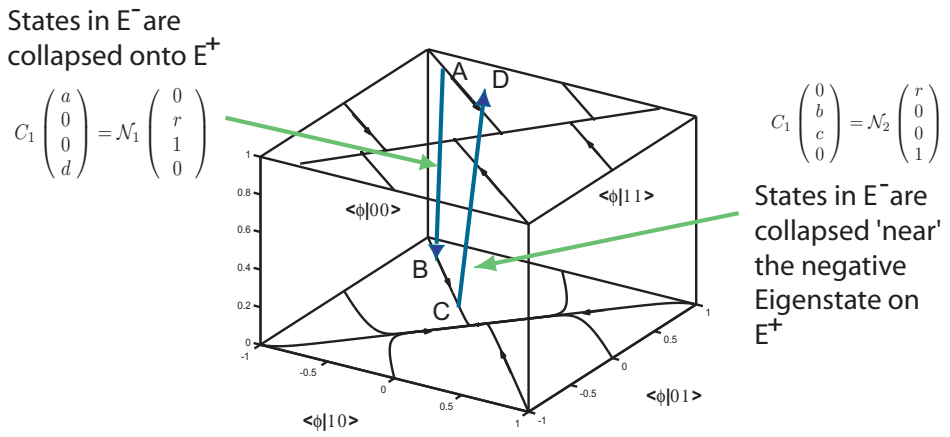

Figure 5.7: Quantum collapses cause the state of the system to oscillate between E^+ and E^-. A state that starts in E^+ (point A) will ultimately collapse onto E^- (point B). Here, it undergoes free evolution to some point C, at which point it will undergo a second collapse to D. Point D is very close to the unstable eigenstate, and thus the state is likely to emit another photon and repeat the A-B-C-D cycle.

Proposition 4 (Equivalence of Quantum Collapses and Measurement) *The probability that a state $|\phi\rangle$ undergoes an eventual quantum collapse is exactly equal to the probability that $|\phi\rangle$ is projected onto the span of the unstable eigenstates of free evolution. Explicitly*

$$P(\text{eventual collapse}) = \sum_{j:\lambda_j \neq 0} |\langle |\phi|\phi_j\rangle\rangle|^2. \tag{5.28}$$

For example, in the current system of interest, any state on E^- is unstable, while a state on E^+ will collapse with probability $|\langle\phi|\phi_2\rangle|^2$ (See Figure 5.6). It is these collapses that couple the two otherwise independent subspaces. The next step is to determine the effect of these collapses.

E^+ and E^- evolve independently under free evolution, while the collapse operators couple the two subspaces. Specifically, the meticulous reader can verify that

$$C_1 \begin{pmatrix} c_{11} \\ 0 \\ 0 \\ c_{00} \end{pmatrix} = \mathcal{N}_1 \begin{pmatrix} 0 \\ -(c_{11} - rc_{00}) \\ c_{11} - rc_{00} \\ 0 \end{pmatrix}, \qquad C_1 \begin{pmatrix} 0 \\ c_{10} \\ c_{01} \\ 0 \end{pmatrix} = \mathcal{N}_2 \begin{pmatrix} r \\ 0 \\ 0 \\ -1 \end{pmatrix}, \tag{5.29}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are constants of normalization. Specifically, this leads to the observations:

- Any state in E^+ collapses onto a state 'close'[4] to $|\phi_4\rangle$ in E^-. Furthermore, this state is always unstable.

[4]The use of 'close' here implies that the two are approximately equal provided r is close to 1.

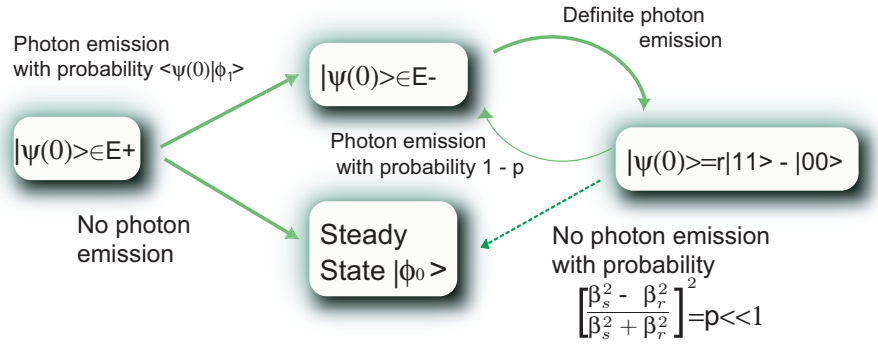

Figure 5.8: Flowchart of the collapse processes in the Cascaded Qubit System. Note that in the limit of resonance, $p = 0$, and the quantum cycle becomes completely stable. (The equation for p will be derived when we treat the cycles analytically.)

- Any state in $|E^-\rangle$ collapses onto a state 'close' to the unstable eigenstate $|\phi_2\rangle$ in E^+ which is 'usually'[5] unstable.

These observations agree with the numerical results obtained earlier, where we concluded that each quantum collapse must result in a shift between E^+ and E^-.

The most important feature is that the resulting state after a quantum collapse from E^- is *independent* of the original state in E^-. More precisely, any state in E^- will be collapsed into the exact same state on E^+ (see Figure 5.7)

Physically, this corresponds to the fact that the system is forced into a specific state after a photon emission, much in the same way as a two level atom is forced into the ground state after spontaneous emission [36]. This is a key reason that quantum trajectories show such a simple cyclic behavior of two jumps.

Suppose our system begins somewhere within E^+ and emits a photon, then we are guaranteed a second emission back to E^+ due to the instability of any resulting state in E^-. The second emission brings it back to $r|11\rangle - |00\rangle$, which is very close to the unstable eigenstate $|\phi_2\rangle = |11\rangle - r|00\rangle$. At this point, there is a finite probability that the state is actually projected onto the steady state $|\phi_1\rangle$. However, provided r is close to 1, Proposition 4 states that this probability will be small. In most occurrences, the system will emit a third photon and repeat the cycle.

Hence given an arbitrary near resonant system, we would expect a large number of such oscillations. Eventually, after some even number of photons have been emitted, the system will settle onto the true steady state. This oscillatory behavior is exactly the one reflected in the first graph of Figure 5.4. We can summarise the behaviour in a convenient flowchart (Figure 5.8).

[5] Again, 'usually' is used in the sense that the probability of collapse approaches 1 as $r \to 1$.

All of the above analysis is done on the proviso that our system did indeed emit a photon to begin with. However, such an occurrence is guaranteed only when the system was already prepared in $|\phi_2\rangle$. In all other cases, there will only be a non-unit probability of initial collapse. This leads us to an interesting phenomenon, where a system can either settle to steady state immediately with no photon emission, or oscillate wildly for a huge number of cycles. We will save this idea for further exploration.

5.4.4 Quantum Cycles at Resonance

With a full qualitative theory of trajectory dynamics in the Cascaded Qubit System, we can now look directly at the case of resonance. Here, we seek to answer one of the questions that motivates this entire discussion, what occurs when the system fails to converge to steady state?

In the case of resonance, the collapse conditions simplify further

$$C_1(E^+) = |\phi_4\rangle, \qquad C_1(E^-) = |\phi_2\rangle. \tag{5.30}$$

That is, if the system is in state $|\phi\rangle$ such that $|\phi\rangle \in E^+$ ($|\phi\rangle \in E^-$), the detection of a photon guarantees that the state has collapsed to $|\phi_4\rangle$, ($|\phi_2\rangle$). The states after emission are now both independent of the original state.

$|\phi_2\rangle$ and $|\phi_4\rangle$ are both unstable eigenstates of free evolution. Thus, the oscillatory behaviour mentioned before becomes completely stable. Once a photon is emitted from the system, the state of the system is collapsed onto one of the two unstable eigenstates, and it remains trapped in these states.

For example, consider a Cascaded Qubit System prepared somewhere in E^+. Suppose at some time t, a photon is detected. We automatically know that the system has undergone collapse and $|\phi\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) = |\phi_4\rangle$. Thus the system will continue to emit photons forever, as the state of the system oscillates between the two Bell states, $|\phi_4\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ and $|\phi_2\rangle = \frac{1}{\sqrt{2}}(|11\rangle - |00\rangle)$.

Now suppose that no photon is detected after an extended period of time, then Figure 5.5 predicts that the state $|\phi\rangle$ will be projected onto the stable Bell State, $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$. This state has no chance of photon emission, and thus the system remains dark for all time. Hence the Cascaded System acts essentially as a measurement process, differentiating between the Bell states $\frac{1}{\sqrt{2}}(|11\rangle \pm |00\rangle)$.

A plot of a typical trajectory is displayed in Figure 5.9. The quantum cycle is self-evident as the state of the system oscillates between $|\phi_2\rangle$ and $|\phi_4\rangle$.

Using simple qualitative techniques of phase portraits and stochastic jumps, we can gain a clear insight onto how each individual trajectory evolves. It is apparent now that the behaviour of each individual trajectory could not be more different from that of the ensemble average. In the limit of resonance, the lack of convergence to steady state is actually caused

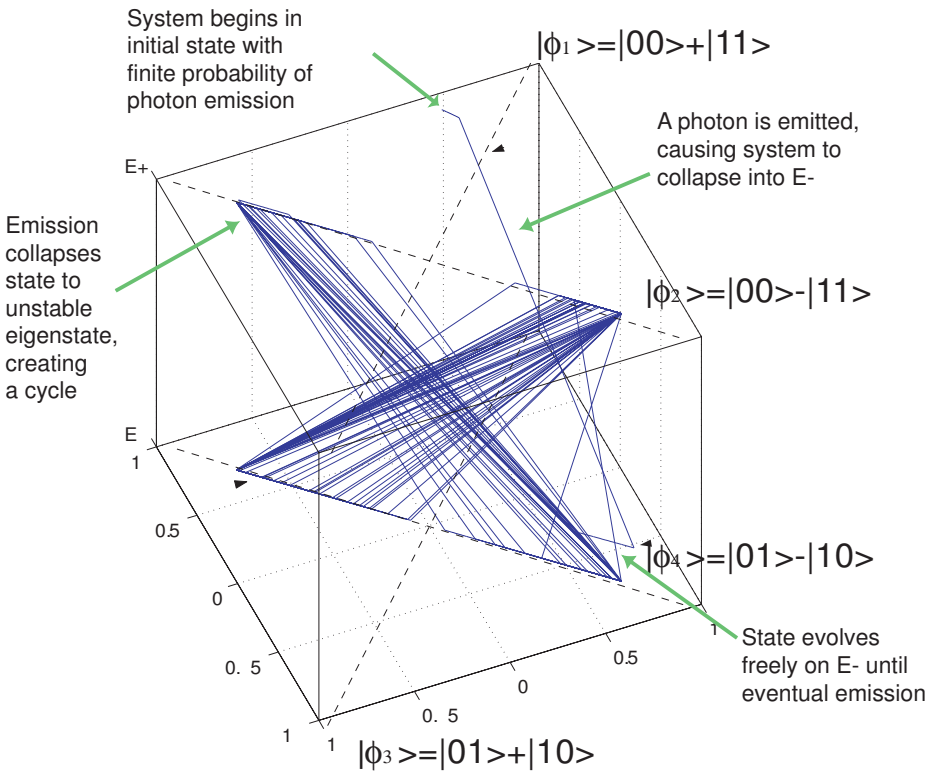

Figure 5.9: A plot of a typical trajectory that displays clear cyclic behavior. Note that this is a stable stochastic cycle between two fully entangled Bell states.

by the fact that all states within the ensemble are trapped in a periodic cycle, and ironically, neither of the states within the cycle correspond to the predicted steady state.

5.5 Analytical Treatment of Quantum Cycles

The evolution of each quantum trajectory can be understood so well by qualitative methods that an analytic solution for such trajectories more appear almost redundant at a conceptual level. However, it is still necessary for specific quantitative predictions and is included for the sake of completeness.

We will begin with a formal solution to the differential equation that governs the free evolution of the system (5.23).

$$\begin{pmatrix} c_{11} \\ c_{10} \\ c_{01} \\ c_{00} \end{pmatrix} = \begin{pmatrix} \frac{1}{r^2+1}\left(\left[r^2 c_{11}(0) + r c_{00}(0)\right] + \left[c_{11}(0) - r c_{00}(0)\right] e^{-2(1+r^2)t}\right) \\ \frac{1}{2}\left(\left[c_{10}(0) + r c_{01}(0)\right] e^{-(r-1)^2 t} + \left[c_{10}(0) - r c_{01}(0)\right] e^{-(r+1)^2 t}\right) \\ \frac{1}{2r}\left(\left[c_{10}(0) + r c_{01}(0)\right] e^{-(r-1)^2 t} + \left[-c_{10}(0) + r c_{01}(0)\right] e^{-(r+1)^2 t}\right) \\ \frac{1}{r^2+1}\left(\left[r c_{11}(0) + c_{00}(0)\right] + \left[-r c_{11}(0) + r^2 c_{00}(0)\right] e^{-2(1+r^2)t}\right) \end{pmatrix} \tag{5.31}$$

As $t \to \infty$, $|\phi\rangle \to \frac{1}{\sqrt{1+r^2}}(r|11\rangle + |00\rangle)$, which is in agreement with our steady state analysis. However, for the resonant case where $r = 1$, it becomes apparent that this steady state is no longer unique, and therefore the long term behaviour of the system can be highly dependent on initial starting conditions.

5.5.1 Evolution of the Quantum Cycle

We first treat the dynamics of the Quantum Cycle itself. Thus, let us consider the evolution of a state that is already on the cycle. That is

$$|\phi(0)\rangle = |\phi_2\rangle = \frac{1}{\sqrt{1+r^2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -r \end{pmatrix} \in E^+, \tag{5.32}$$

which corresponds to an eigenstate of H_{eff} with eigenvalue $2i(1+r^2)$, thus

$$|\phi(t)\rangle = e^{-2(1+r^2)t}|\phi(0)\rangle \tag{5.33}$$

Noting that the probability of collapse is given by:

$$p_c = 1 - \lim_{t\to\infty} |||\phi(t)\rangle||^2, \tag{5.34}$$

we have $p_c = 1$, and thus the state will certainly jump at some time, say t_1 such that $|\phi(t_1)\rangle \in E^-$. Now we break down the evolution into cases:

- If $r \neq 1$, E^- is globally unstable, and the system will undergo a second quantum collapse at time t_2.
- If $r = 1$ then $|\phi(t_1)\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ which has eigenvalue -4 and hence is unstable. Hence the system will also undergo a second quantum collapse at time t_2.

In either case, a second photon detection is inevitable, and therefore a second collapse at some time t_2 is guaranteed. From (5.42),

$$|\phi(t_2)\rangle = \frac{1}{\sqrt{1+r^2}} \begin{pmatrix} r \\ 0 \\ 0 \\ -1 \end{pmatrix}. \tag{5.35}$$

While this is not the state that we started with, the two states are very similar, as r is close to 1, . Thus, there is a high probability that a third collapse is observed, and the system undergoes another two jump cycle as described above. We can compute this probability explicitly by considering

$$|\phi(t_2+t)\rangle = \frac{1}{(1+r^2)^{\frac{3}{2}}} \begin{pmatrix} r(r^2-1) + 2re^{-2(1+r^2)t} \\ 0 \\ 0 \\ (r^2-1) - 2r^2e^{-2(1+r^2)t} \end{pmatrix}. \tag{5.36}$$

Thus, the probability of not undergoing another collapse is given by

$$q_c = \lim_{t\to\infty} |||\phi(t_2+t)\rangle||^2 = \left[\frac{r^2-1}{r^2+1}\right]^2 = \left[\frac{\beta_s^2 - \beta_r^2}{\beta_s^2 + \beta_r^2}\right]^2. \tag{5.37}$$

Provided that the system is not resonant, this oscillation will eventually end when no photons are observed, and the state settles to the steady state predicted by steady state analysis.

For near resonant systems, $q_c \ll 1$, and hence there is a very high chance per two jump cycle that another collapse will occur, causing the cycle to repeat itself and the emission of two more photons. In the limit of resonance, the system will be locked in this two jump cycle for all time.

Therefore, we can consider this oscillation of jumps as a meta-stable periodic steady state, in which the system will stay for an extended amount of time. This coincides with the predictions made by numerical analysis, and demonstrates behaviour that could not be observed via conventional numerical integration of the master equation.

5.6 Quantum Cycles in the Presence of Coupling Loss

In this section, we will consider the robustness of the cycles to (small) coupling loss, i.e, when $\epsilon < 1$. In this case, we define $\Delta = 1 - \epsilon \ll 1$, and show that such a system not only conserves the original quantum cycle, but also generates a second stable cycle.

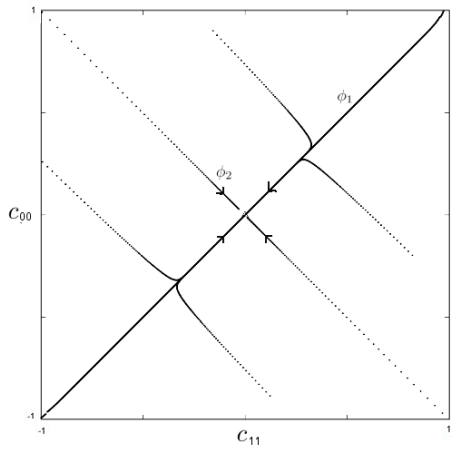

Figure 5.10: Phase portrait of Free Evolution Dynamics in the E^+ plane when $r = 1.0$, $\epsilon = 0.9$. Note that the directions of the eigenvectors are not affected by the non-optimal coupling. However, the norm of $|\phi_1\rangle$ is no longer conserved. The phase portrait for the E^- plan is identical to that above, switching $|\phi_1\rangle$ for $|\phi_3\rangle$ and $|\phi_2\rangle$ for $|\phi_4\rangle$.

5.6.1 System Dynamics

Using the matrix representation of the operators, we can show that the free evolution of the system, with non-optimal coupling, is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}|\phi\rangle = \begin{pmatrix} -2 & 0 & 0 & 2r\sqrt{\epsilon} \\ 0 & -(1+r^2) & 2r^2\sqrt{\epsilon} & 0 \\ 0 & 2r^2\sqrt{\epsilon} & -(1+r^2) & 0 \\ 2r\sqrt{\epsilon} & 0 & 0 & -2r^2 \end{pmatrix} |\phi\rangle, \tag{5.38}$$

which can again be separated into independent evolution on E^+ and E^-.

In order to keep the algebra tractable, we will limit a rigorous analysis to the case where $r = 1$. Specifically, we find eigenvectors $\phi_1 = (1,0,0,1)^T$, $\phi_2 = (-1,0,0,1)^T$, $\phi_3 = (0,1,1,0)^T$, $\phi_4 = (0,1,-1,0)$ with corresponding eigenvalues $\lambda_1 = -4$, $\lambda_2 = -\Delta$, $\lambda_3 = -4$, $\lambda_4 = -\Delta$. where we have used the small coupling loss approximation

$$|1-\epsilon| \ll 1 \Rightarrow \sqrt{\epsilon} = \sqrt{1-\Delta} \approx 1 - \frac{\Delta}{2} \tag{5.39}$$

The dynamics on E^+ and E^- are demonstrated by the phase portrait in Figure 5.10. The largest difference is that $|\phi_1\rangle$ is no longer a steady state of the system. The extra loss Δ guarantees that a state initialized at $|\phi_1\rangle$ will undergo a quantum collapse at some point. Thus, the equilibrium is no longer a pure state in agreement with [27].

Quantum Collapse Operators

The system can now undergo quantum collapse via two different collapse operators, C_1 and C_2. C_1 corresponds to the emission of a photon from the system, as before, while C_2 corresponds to the absorption of a photon from the non-optimal coupling between the two cavities. In matrix representation

$$C_1 = \sqrt{2}\begin{pmatrix} 0 & -\sqrt{\epsilon}r & r & 0 \\ -\sqrt{\epsilon} & 0 & 0 & r \\ 1 & 0 & 0 & -\sqrt{\epsilon}r \\ 0 & 1 & -\sqrt{\epsilon} & 0 \end{pmatrix}|\phi\rangle, \tag{5.40}$$

$$C_2 = \begin{pmatrix} 0 & r & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 1 & 0 \end{pmatrix}|\phi\rangle. \tag{5.41}$$

Consider, first, the effect of a photon emission due to C_2,

$$C_2\begin{pmatrix} a \\ 0 \\ 0 \\ b \end{pmatrix} = \mathcal{N}\begin{pmatrix} 0 \\ a \\ rb \\ 0 \end{pmatrix}, \qquad C_2\begin{pmatrix} 0 \\ a \\ b \\ 0 \end{pmatrix} = \mathcal{N}\begin{pmatrix} ra \\ 0 \\ 0 \\ b \end{pmatrix}. \tag{5.42}$$

In particular, for $r = 1$, C_2 directly maps each vector in E^+ onto the coinciding vector in E^-. Since the free evolution dynamics on each subspace is identical, the action of C_2 on the long term behavior of the system can be effectively disregarded.

5.6.2 Quantum Cycles of the Generalized System

Stability of the First Quantum Cycle

Let us consider a resonant system with non-optimal coupling. We first observe that the relations

$$C_1|\phi_2\rangle = \mathcal{N}|\phi_4\rangle \qquad C_1|\phi_4\rangle = \mathcal{N}|\phi_2\rangle \tag{5.43}$$

still hold. Therefore, the effect of a photon emission is identical to that of an optimally coupled system. Also, (5.42) shows that the action of C_2 is identical to that of C_1 for states within the quantum cycle. As the two collapse operators have the same effect, it is clear that inefficient coupling does not destabilize systems that exhibit such behavior.

The Second Quantum Cycle

Recall that $|\phi_1\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$ is no longer a steady state of the system. Since $\langle\phi_1|C_1^\dagger C_1|\phi_1\rangle = 0$ from (5.40), any collapse from such a state must be caused by inefficient coupling. That is, a collapse via the operator C_2.

Consider a system that starts in state $|\phi_1\rangle$, it must lose a photon due to inefficient coupling at some time t_1, so that

$$|\phi(t_1)\rangle = \frac{C_2|\phi_1\rangle}{\|C_2|\phi_1\rangle\|} = |\phi_3\rangle \tag{5.44}$$

But $|\phi_3\rangle$ has the identical evolution dynamics on E^- as $|\phi_1\rangle$ on E^+. That is, $\langle\phi_3|C_1^\dagger C_1|\phi_3\rangle = 0$ still holds, and hence there can be no photon emission. However as $|\phi_3\rangle$ is unstable, it must still undergo a quantum collapse, and that collapse must again be due to inefficient coupling. Thus, at some time t_2, a second photon will be lost, and the system is restored to the state $|\phi_1\rangle$.

Thus the introduction of coupling inefficiency converts the original steady state into a quantum cycle completely orthogonal to the first one, oscillating between $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle$ and $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$.

Both cycles are stable and attractive, and offer two possibilities for the long term behaviour of the system. Exactly which of the two cycles a particular system will evolve into will depend on the initial state of the system.

5.6.3 Non-Optimal Coupling in Experiments

The question remains however, as to how this loss in coupling will affect experimental attempts to detect quantum cycles. While it is indeed true that the inclusion of non-unit coupling efficiency does not destroy the existence of such cycles, we need to remember that the second detector that measures photons lost in transmission cannot be implemented experimentally. This is not quantum cryptography, and it would be impractical to consider the experimenter to have divine powers at his or her disposal, like 'Eve'.

Suppose a system was to begin in $|\phi_2\rangle$, then in the case of optimal coupling, we can be certain that the state of the system will again be in $|\phi_2\rangle$ after any even number of photon detections. However, in the presence of coupling loss, a photon could have escaped undetected during this time and hence our certainty about the state of the system is lost.

For example, consider the probability of photon loss being around 1% per cycle, then after one cycle, the state of our system will be a mixed state of the form $0.99|\phi_4\rangle\langle\phi_4| + 0.01|\phi_4\rangle\langle\phi_4|$, and without the measurement history in our possession, we can no longer unravel the system exactly to extract all of its properties. Thus, for realistic experiments, these quantum cycles will not last forever, but can certainly be observed for a number of cycles.

5.7 Non-Causal Behaviour

> " We must believe in free will. We have no choice. "
> –*Isaac B. Singer*

The existence of quantum cycles opens the door to interesting phenomena, all of which could

not be predicted through numerical integration of the master equation. The core reason here is simple, a master equation is an equation for the ensemble average, and in considering an ensemble average, we lose the information about each individual.

Classical Newtonian mechanics was a logical science, with a simple, satisfying statement. The behaviour of a system is completely characterized by its initial conditions. If you knew the initial conditions precisely, you could characterize the evolution of the system precisely. Yet, in quantum mechanics, the measurement postulate is far from causal [37]. Despite this, it retains some illusion of causality due to the deterministic nature of free evolution.

Consider now the cascaded qubit system. In this section, we will show that given the same starting condition, this system can exhibit two completely different types of behaviour. On one hand, it can settle directly to the steady state, while on the other, it can continue to emit photons for an extended period of time. In fact, our treatment will be very general, and apply to cases of non-optimal coupling.

Consider a system that is initialized in some $|\phi(0)\rangle \in E^+$. We can write

$$|\phi(0)\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle \tag{5.45}$$

so that, for $r = 1$

$$|\phi(t)\rangle = c_1 e^{-\Delta t}|\phi_1\rangle + c_2 e^{-4t}|\phi_2\rangle \tag{5.46}$$

under free evolution. Using the collapse probabilities

$$\left\langle \phi_1|C_1^\dagger C_1|\phi_1\right\rangle = 0 \qquad , \left\langle \phi_2|C_1^\dagger C_1|\phi_2\right\rangle = 8, \tag{5.47}$$

$$\left\langle \phi_1|C_2^\dagger C_2|\phi_1\right\rangle = 2\Delta \qquad , \left\langle \phi_2|C_2^\dagger C_2|\phi_2\right\rangle = 0 \tag{5.48}$$

And that all cross correlations are identically zero, we can easily compute the collapse probabilities during free evolution as follows

$$P_1(t) = \frac{\left\langle \phi(t)|C_1^\dagger C_1|\phi(t)\right\rangle}{\langle\phi(t)|\phi(t)\rangle} = \frac{8|c_2|^2}{|c_1|^2 e^{(8-2\Delta)t} + |c_2|^2}$$

$$P_2(t) = \frac{\left\langle \phi(t)|C_2^\dagger C_2|\phi(t)\right\rangle}{\langle\phi(t)|\phi(t)\rangle} = \frac{2\Delta}{|c_1|^2 + |c_2|^2 e^{-(8-2\Delta)t}}$$

Let us now consider the probability $\mathcal{P}_1$, that a photon is detected from the output cavity. That is, the probability that the system undergoes quantum collapse via the operator C_1.

$$\frac{dQ}{dt} = -\left(\frac{d\mathcal{P}_1}{dt} + \frac{d\mathcal{P}_2}{dt}\right) \tag{5.49}$$

$$\frac{d\mathcal{P}_i}{dt} = Q(t)P_i(t) \qquad i = 1, 2 \tag{5.50}$$

Where $\mathcal{P}_i(t)$ is the probability that the system undergoes quantum collapse via operator C_i and $Q(t)$ the probability that no photons have been emitted after time t. Solving, we find

$$\mathcal{P}_1 = \lim_{t\to\infty} \int_0^t P_1(t) \langle\phi(t)|\phi(t)\rangle \, dt = |c_2|^2 \tag{5.51}$$

which is exactly the probability one would expect for $|\phi(0)\rangle$ to be projected onto the Bell state $|\phi_2\rangle$.

Thus, the detection of a photon emitted by the system is physically identical to a measurement of the system with respect to the Bell state basis vectors $|11\rangle \pm |00\rangle$. We also note that our treatment includes non-optimal coupling, and thus the measurement analogy holds in realistic systems.

Now, let us consider how the system evolves depending on whether a photon is emitted from the system or not

No Photon Emission

First suppose that no photon was ever detected from the system in *Detector 1*, then (5.46) implies that

$$|\phi(t)\rangle = e^{-\Delta t}|\phi_1\rangle \tag{5.52}$$

when t becomes large. Thus the absence of a photon emission allows us to determine that the system has evolved into the $|\phi_1\rangle, |\phi_3\rangle$ quantum cycle. That is, the system will fluctuate between $|\phi_1\rangle$ and $|\phi_3\rangle$, losing photons via inefficient coupling with each jump. In the event that coupling is perfect, the system settles into the pure state $|\phi_1\rangle$ (since $|\phi(0)\rangle \in E^+$).

Initial Photon Emission

Alternatively, suppose a photon is emitted from the system at time t_1, then

$$|\phi(t_1)\rangle = \mathcal{N}C_1 \begin{pmatrix} a \\ 0 \\ 0 \\ b \end{pmatrix} = \frac{1}{\sqrt{(a^2+b^2)(1+\epsilon) - 4ab\sqrt{\epsilon}}} \begin{pmatrix} 0 \\ -a\sqrt{\epsilon}+b \\ a - b\sqrt{\epsilon} \\ 0 \end{pmatrix} \approx \frac{1}{\sqrt{2}(a-b)} \begin{pmatrix} 0 \\ -a\sqrt{\epsilon}+b \\ a - b\sqrt{\epsilon} \\ 0 \end{pmatrix}$$

which will lead to a second photon emission with probability

$$\begin{aligned} 1 - \langle\phi_3|\phi(t_1)\rangle &\approx 1 - \frac{(a+b)(1-\sqrt{\epsilon})}{\sqrt{2}(a-b)} \\ &\approx 1 - \frac{\Delta a(a+b)}{2\epsilon} = 1 - \Delta\frac{a+b}{4(a-b)} \end{aligned} \tag{5.53}$$

As $\Delta \ll 1$, and $a \neq b$ is guaranteed by the fact that $a = b$ corresponds to the case where there would be no C_1 collapse, the emission of one photon will almost certainly imply that the system will continue to emit photons, and settle into the $|\phi_2\rangle, |\phi_4\rangle$ cycle.

The two types of evolution could not be more different. In the case of no initial photon emission, no light ever escapes the second cavity, and our system is dark. However, if just one photon is emitted, the system is trapped in the emission cycle, and will continue to emit photons for a very extended period of time. And all this spontaneous activity, without any

explicit action from the experimenter! The baffling fact is that two types of behaviour can be produced by the same system, under the same initial conditions, without any prior causes. Indeed, it is behaviour like this that some people choose to call free will [50].

Of course, the fact that a output channel exists leads leads to an implicit measurement. That is, we induce spontaneous behaviour within the system when we extract classical information from the environment, due to the quantum correlations it shares with the system. When a system chooses whether or not to emit a photon, it is equivalent to a measurement made by the experimenter with respect to the Bell state basis. The truly peculiar characteristics of quantum mechanics again manifests themselves.

If a cascaded qubit system was placed in an open environment, with no photon detector to 'hear' it, does it oscillate? Such philosophical questions could form entire essays, which we will leave to the reader, or a particularly enthused summer student. Suffice to say, without the use of trajectories, such behaviour would be completely invisible, for the ensemble average would have hidden this spontaneous activity, and instead produce a misleading, deterministic solution to the master equation.

As a practical application, the procedure noted above could very well be used to perform measurements in the Bell state basis, as required in many quantum information protocols such as teleportation [38]. The advantage here is that a perfect detector is unnecessary, as the system will either emit no photons, or many, many photons.

5.8 Entanglement of Evolution

Finally, we conclude this chapter by coming full circle to the question that launched this discussion. How did entanglement evolve throughout the system? Using standard master equation techniques, this question could not be answered exactly due to the ambiguity of mixed states. Instead, one had to resort to approximate estimates, such as the fidelity of the current states with respected to the Bell state that we expected the system to evolve to.

The difficulty with such measurements is that there is more than one Bell state with perfect entanglement, and hence decreasing fidelity does not necessarily imply decreasing entanglement. Further, all such measures are applied to the average of quantum trajectories, and as we have shown in Chapter 3, the entanglement of the average is not the same as the average entanglement. As such, these methods implicitly assume that at any point in time, the trajectories themselves are roughly similar, and evolve relatively smoothly. Yet, a brief look at Figure 5.9 will show that this is far from the truth, the trajectories in our system oscillated between two completely orthogonal states, and at any one time, two trajectories could be completely orthogonal to each other.

In the trajectory formulation, entanglement of evolution can be solved exactly. We do not even need to resort to calculating the entanglement of individual trajectories, and then taking the average. We already know that in the case of resonance, the trajectories are oscillating between two perfectly entangled Bell states, $|\phi_2\rangle$ and $|\phi_4\rangle$. Therefore, we are able up observe a very strong conclusion

Proposition 5 (Entanglement within the Cascaded Qubit System) *The entanglement of the cascaded qubit system is* 1 *for any time* t *after the initial photon emission at resonance in the presence of a perfect photon detector.*

Quantum trajectory theory has given us an appropriate unravelling of the density operator and shown us that the constituents of the mixed ensemble were actually bell states. That is, it is unnecessary to wait for the steady state to prepare a perfectly entangled state. We can simply stop the evolution of the system after any number of photon detections, and be guaranteed to have a perfectly entangled state within the system.

5.8.1 The Measurement Entanglement Paradox

Nothing in quantum mechanics, however, is so simple. One has to remember that the resulting unravelling of the master equation into Bell states was caused by a specific choice of measurement process. In this case, it turned out that one of the simplest choices of measurement scheme, namely photon detection, was capable to inducing the system into a stochastic cycle of perfectly entangled states. However, we must stress that the choice of measurement is not unique, and different choices of measurements will give rise to different trajectories, and possibly different values of entanglement during evolutions.

The key difference here, between an arbitrary unravelling, is that the knowledge of the measurement history of the detector allows the experimenter to induce the pure state of the system, and thus extract the entanglement. This demonstrates that a passive, continuous measurement process is capable of generating entanglement within a system that it did not directly interact with.

This is, in essence, bizarre. In classical physics, it is akin to taking photographs of a cannon ball to alter its trajectory, or believing that a supernova occurred in a distant galaxy because someone pointed a telescope at it. Yet, entanglement in a system can be created simply by looking at it? Or should we argue that it was always there in the first place, and that it just cannot be accessed? We will defer a lengthy discussion to this question to our concluding remarks in Chapter 8.

Regardless of your interpretation, the cascaded qubit system shows that continuous measurements are a very promising method to create entanglement within open systems.

Chapter 6

Parity Entanglement of Qudits

"Any theorem can be fitted onto an arbitrarily small piece of paper if you're sufficiently obscure."
"Proof of Theorem. 6.2 is trivial from Theorem. 6.9"
-*Anonymous lecturer, Cambridge University.*

6.1 Multi-Dimensional Computation

The classical computer is based on the logic of bits; every data register is either a 1 or a 0. A classical processor operates on statements that are either absolutely TRUE or absolutely FALSE, and it is difficult to imagine that the formalism could have been completely different.

In particular, binary computation is not the only form of computation. In fact, the first analogue computers, such as the abacus, and difference machine, were all based on the decimal system. This is natural, given the prevalence of decimal representation of numbers in the international community. It was only in 1939, when Atanasoff had the epiphany that it was more efficient to represent binary over decimal numbers digitally that the ideas of binary arithmetic were realized [39].

But why are binary numbers more efficient? And are there times where multi-dimensional logic would be more appropraite?

6.1.1 Optimal dimensionality of computation

Arithmetic need not be done in base 2 or base 10, it can be done in any base. However, is there an optimal basis which minimizes some sort of physical resource? Intuitively there has to be, for let us simply consider memorizing some large number, say 1234.[1]

[1]Here, we can use the standard alphanumerical representation of numbers greater than 10 in hyper-decimal basis representation, i.e: A denotes 10, B denotes 11 etc. Also $\mathcal{Z}$ denotes 1234.

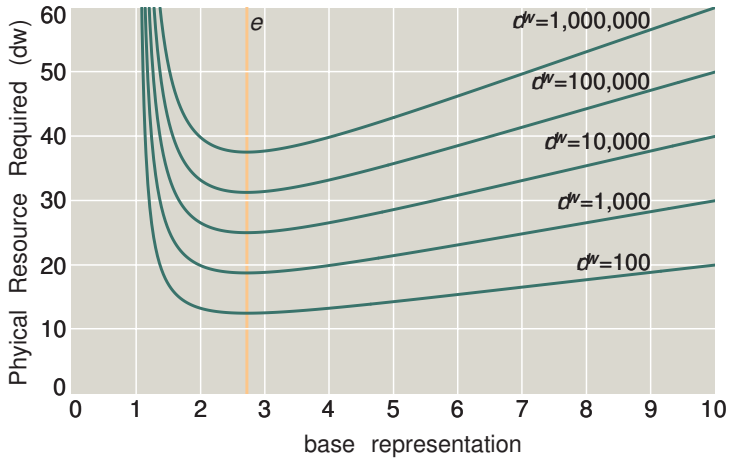

Figure 6.1: A graph of the physical resources required to store specific numbers for each given basis. The asymptote at $b = 1$ signifies that it is impossible to represent numbers in unary. Note that for all cases, the representation in base 3 uses the least physical resource.

- **Base 2:** 10011010010
- **Base 3:** 1200301
- **Base 16:** $4D2$
- **Base 36:** YA
- **Base 1234:** $\mathcal{Z}$

In one limit (Base 2), we have a long chain of numbers that we need to remember; in the other limit we require an enormous number of states to represent each individual digit. Neither appears particularly optimal, and one can reason the optimal representation should lie somewhere in between.

Suppose that we assume that it takes the same amount of physical resource to store a physical state. That is, a two digit, base 2 number costs 4 units of space, while a one digit base 3 number costs 3 units of space, etc, then the total resource required to store a number $n = d^w$ is exactly dw, where d is the dimensionality of each digit and w is the number of digits required to represent the number n. Therefore, finding the optimal base d is reduced to minimizing dw subject to the constraint $n = d^w$.

Physically this problem is no different to the problem of optimizing a telephone auto-responder, where the aim is to minimize the number of choices a poor customer needs to listen to when he needs to make a choice between $n = d^w$ options. where d is now the number of choices each time the customer presses a button, and w the number of buttons he needs to press [40].

This is a simple calculus problem with an analytical result. It turns out that the optimal basis is exactly $e = 2.718\ldots$. American Scientist demonstrates this nicely via Figure 6.1[40].

As non-integer bases are rather inconvenient to deal with, one can show that the optimal basis is actually 3.

While there have been numerous experimental ternary computers, such as the experimental "Setun" in Moscow State University during the Cold War [41], ternary computers never entered the mainstream.

The reason for this is one of practicality; a system of two states can be easily represented by the existence or non-existence of a current, a simple switch that is easy to measure and implement. In comparison, a system that has to discern three different voltages is much trickier and much more prone to errors. The increase in technical difficulty of introducing a third state is certainly not a linear increase, that is, a trit took far more than 150% the physical resource needed for a bit.

6.1.2 Higher Dimensional Quantum Computation

Quantum computers have very different physical limitations. For example, one of the primary difficulties is the limited number of qubits that can be coupled simultaneously. This could potentially shift the preferred dimensionality in the other direction, so that we use higher dimensional qudits to minimize the number of them that need to be coupled.

The advantages of qudits can be easily understood in terms of entanglement; whereas the maximal entanglement of two qubits is 1, two qudits can have mutual entanglement of up to $\lg d$. Since entanglement can be freely converted between different forms, the capacity to entangle higher dimensional systems has the capacity of yielding greater resources for quantum communication protocols. For example, Karimipour et al. [42] have presented a quantum key distribution protocol utilizing qudits whose security improves with the dimensionality of the system.

In the previous chapter, we have already demonstrated that the cascaded qubit system is a promising method for generating entanglement between qubits. Could it also be used to generate entanglement between qudits? It is evident that when multiple atoms are placed within each optical cavity, a qudit can be formed out of the combined angular momentum states. Can these states be entangled by the cascaded cavity system, and if so, how well?

In this chapter, we will extend the steady state results of the cascaded qubit system [27] to apply to cascaded systems of many atoms.

6.2 Representations of the Many-Atom System

The general version of the Cascaded Cavity System, with multiple atoms per cavity, is quite a difficult system to treat. To tackle the problem, we first introduce some notation which will help simplify the system both analytically and conceptually. Recall from Chapter 4 that

the cascaded cavity system can be described by the master equation

$$\begin{aligned}\dot{\rho} &= \sum_{i=1,2}\left(2R_i\rho R_i^\dagger - R_i^\dagger R_i\rho - \rho R_i^\dagger R_i\right) \\ &\quad +2\sqrt{\epsilon}(\rho R_1^\dagger R_2 - R_2\rho R_1^\dagger + R_2^\dagger R_1\rho - R_1\rho R_2^\dagger) \qquad (6.1)\end{aligned}$$

where

$$R_k = \beta_{r,k}^* S_k + \beta_{s,k}^* S_k^\dagger \qquad (6.2)$$

is the operator that can be associated with cavity emission and S_k are standard ladder lowering operators in the angular momentum representation.

From previous analysis, it was demonstrated that the loss in coupling introduces extra classical uncertainty into the system, but does not serve to severely alter the behavior of the system itself. Hence, for the purposes of simplicity, we will limit ourselves to the case where coupling is optimal ($\epsilon = 1$). The major motivation for this is that the steady state is pure only when there is no auxiliary loss, and we would like to exactly characterize how the steady state entanglement is affected by increasing atom numbers.

Analogous to the cascaded qubit system, we can rewrite the master equation in the trajectory formalism

$$\dot{\rho} = i\left[\rho, H_{eff}\right] - \frac{1}{2}C_1\rho C_1^\dagger, \qquad (6.3)$$

Where we define

$$\begin{aligned}H_{eff} &= i\sqrt{\epsilon}(R_2^\dagger R_1 - R_1^\dagger R_2) - \frac{i}{2}(C_1^\dagger C_1), \qquad (6.4)\\ C_1 &= \sqrt{2}(R_1 - R_2). \qquad (6.5)\end{aligned}$$

So far, nothing new has been introduced. This is a simple generalization of the qubit system, where the Pauli operators have been replaced with angular momentum lowering and raising operators. This elucidates a simple interpretation. In the qubit system, the emission of a photon from a cavity corresponded to the atom within the cavity making a transition between $|0\rangle$ and $|1\rangle$ or vice versa. Now, the emission simply causes one of the n atoms within the cavity to switch states. Since each atom can be treated as if it were a spin-$\frac{1}{2}$ particle, a switch in state simply corresponds to a change of total angular momentum in some fixed direction. From this perspective, it is natural for a cavity emission operator R to consist of a sum of S and $S^\dagger$.

6.2.1 The Angular Momentum Representation

Consider one isolated cavity with n atoms. The most obvious representation for such a state would simply be a product state of all individual atoms. However, such a system would have a Hilbert space whose dimensions grow exponentially with respect to the number of atoms.[2]

Fortunately, symmetry can be used to simplify the problem. One can observe that the dynamics of the system are governed entirely by operators S and $S^\dagger$, which are completely

[2] A system of n atoms would clearly have dimensions 2^n.

invariant with respect to atom interchange. That is, the dynamics of the system does not distinguish between individual atoms, and thus there is no need to describe the state of each atom individually. For a complete characterization of the system, all we need to know is *how many* of the n atoms are in state $|1\rangle$.

In the angular momentum representation, this implies that the total angular momentum $|S|^2$ is conserved. Explicitly, we know that a system of n spin-$\frac{1}{2}$ particles are typically represented by the states

$$|l, m\rangle, \qquad l = \frac{n}{2}, \frac{n}{2} - 1, \ldots, \frac{1}{2}, \qquad m = l, l-1, \ldots, -l \tag{6.6}$$

However, due to the conservation of total angular momentum, provided we start in a state that is symmetric with respect to all atoms, such as $|000\ldots0\rangle$, then the total angular momentum will remain at $l = \frac{n}{2}$ for all time. Thus, the state of the system can be characterized entirely by its angular momentum in the 'z-direction',[3] which allows us to make the following definition.

Proposition 6 (Collective Angular Momentum Representation of Atomic States) *Assume that the cascaded atomic system is initialized in a state that is symmetric with respect to local atom interchange. Let $\mathcal{C}_n$ be the Hilbert Space that represents the space of states for a system with n identical atoms, where*

$$\mathcal{C}_n = \{|j, m\rangle\}, \qquad j = \frac{n}{2} \qquad m = -j, -j+1, \ldots, j-1, j. \tag{6.7}$$

Then $|j, n_{up} - \frac{n}{2}\rangle$ represents the state with n_{up} atoms in the $|1\rangle$ state. A pure state of the cascaded cavity system at any time can be represented by the product space $\mathcal{C}_n \times \mathcal{C}_n$.

This representation is exact, and will be used for general purpose analysis. It reduces the dimensions of an n-atom system to $n+1$, so that the dimensions of the cascaded system only grows quadratically with respect to n. Needless to say, its a nice improvement.

6.2.2 The Schwinger Representation

The state of a many-atom system can be described in a completely different manner if we begin with the assumption that all atoms are identical. This assumption is valid under the same conditions as before, that is, provided the state of the system begins in a symmetric state. Now the state of the atoms can be easily characterized in terms of the atoms in each of the $|0\rangle$ and $|1\rangle$ states.

Each state essentially behaves as a mode, whose energy is determined by the number of atoms in the excited state. Analogous to quantized field modes, we can define annihilation operators a and b that correspond to the removal of an atom from the $|1\rangle$ and $|0\rangle$ states respectively and rewrite the cavity dynamics in terms of these operators (See Figure 6.2).

Recall that the action of S is to switch one atom within the cavity from $|1\rangle$ to $|0\rangle$, that is, it annihilates an atom from the $|1\rangle$ state and creates one in the $|0\rangle$ state. Hence we immediately

[3] Of course, since angular momentum is used here only for its algebraic properties, the z-direction does not actually exist in real space either.

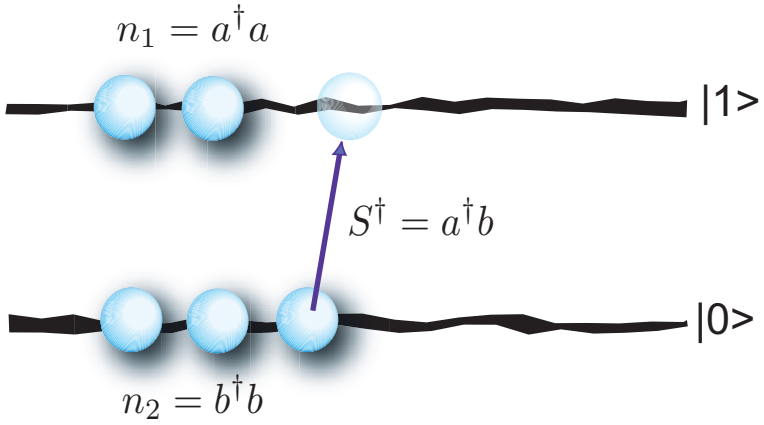

Figure 6.2: The Schwinger representation of the collective atomic system.

have

$$S = ab^\dagger, \qquad S^\dagger = ba^\dagger. \tag{6.8}$$

A quick look will confirm that this is exactly the Schwinger formulation for angular momentum. [4] Thus S and $S^\dagger$ obey all standard angular momentum commutation relations.

While the Schwinger representation is typically a mathematical formality, with the modes that it introduces being unphysical, the situation is actually reversed in the cascaded cavity system. In a sense, the Schwinger representation is a true physical interpretation of the system, while its interpretation as angular momentum is for algebraic convenience. It also turns out that this representation is especially useful when we wish to make approximations to the system.

6.3 Conditions for Steady State

In order to characterize the entanglement of the system at steady state exactly, we first need to solve for the steady state exactly. While such a problem was trivial for the cascaded qubit system, a general solution for the case where each cavity has n atoms is significantly more difficult. In particular, we wish to observe a general pattern for the steady states with respect to n, and hence derive general statements about the conditions under which entanglement is generated.

The obvious requirement for steady state is ato solve the steady state master equation. Suppose, however, that there exists a pure state $|\phi\rangle$ that is a steady state for the system, it is then useful to consider the trajectory formalism. A pure steady state, $|\phi\rangle$ must decouple from the environment, that is, $C_1|\phi\rangle = 0$. In addition, the state must be conserved under

[4] A good summary of the Schwinger representation is available in the advanced quantum mechanics textbook by Sakurai [37].

free evolution with respect to H_{eff}. Thus, $|\phi\rangle$ must satisfy the conditions:

$$C_1|\phi\rangle = 0, \tag{6.9}$$
$$H_{eff}|\phi\rangle = \lambda|\phi\rangle, \tag{6.10}$$

for some λ. However, recall that for a state to be conserved, its norm must not decrease under free evolution. For a decreasing norm signifies that information is being lost to the environment, and will eventually lead to a quantum collapse. Thus, $C_1|\phi\rangle = 0$ implies that $H_{eff}|\phi\rangle = 0$, which allows us to write down a stronger condition for steady state.

Proposition 7 (Steady State Condition) *Suppose $|\phi\rangle$ is the steady state of the cascaded cavity system. Then $|\phi\rangle$ must satisfy*

$$C|\phi\rangle = (R_1 - R_2)|\phi\rangle = 0, \tag{6.11}$$
$$D|\phi\rangle = (R_2^\dagger R_1 - R_1^\dagger R_2)|\phi\rangle = 0. \tag{6.12}$$

Note that we have defined the operators C and D for the sake of algebraic simplicity. While we have deduced this proposition using informal arguments, it can also be proven rigourously.

From $C_1|\phi\rangle = 0$ and $H_{eff}|\phi\rangle = \lambda|\phi\rangle$, it follows that

$$R_1|\phi\rangle = R_2|\phi\rangle \tag{6.13}$$
$$(R_2^\dagger R_1 - R_1^\dagger R_2)|\phi\rangle = i\gamma|\phi\rangle. \tag{6.14}$$

for some $\gamma \in \mathbb{R}$, since $R_2^\dagger R_1 - R_1^\dagger R_2$ is anti-hermitian. Substitution of (6.13) into (6.14) gives

$$(R_2^\dagger R_1 - R_1^\dagger R_2)|\phi\rangle = (R_2^\dagger R_2 - R_1^\dagger R_1)|\phi\rangle = i\gamma|\phi\rangle, \qquad \gamma \in \mathbb{R}. \tag{6.15}$$

Taking the complex conjugate of the above equation, we find

$$(R_2^\dagger R_2 - R_1^\dagger R_1)|\phi\rangle = -i\gamma, |\phi\rangle \qquad \gamma \in \mathbb{R}. \tag{6.16}$$

Comparing this with the previous equation (6.15) immediately gives $\gamma = 0$ and hence $H_{eff}|\phi\rangle = 0$.

Proposition 7 tells us that any pure steady state of the system must reside in the null space of both operators C and D. The problem of solving for the steady state is now reduced to a technical problem in linear algebra.

6.4 Numerical Methods

The first step in tackling a difficult problem is to gain an intuitive feel for the solutions, and there is no better way than by using numerical methods. While such methods cannot shed light on how the solutions occur, or be used to verify limiting cases, they allow us to make conjectures about future analytical calculations. In this section, we present a general numerical method to solve for the steady state of a cascaded cavity system with n atoms in each cavity.

6.4.1 The Algorithm

As with any numerical program, the first step is to convert operators and states into matrices and vectors. The chosen representation is a direct extension of the qubit system. A state in $\mathcal{C}_n \otimes \mathcal{C}_n$ can be written as a vector of $(n+1)^2$ dimensions. Explicitly, we write

$$|\phi\rangle = \sum_{j,k=0}^{n} c_{j,k}|-n/2+j\rangle_1|-n/2+k\rangle_2 = \begin{pmatrix} c_{n,n} \\ \vdots \\ c_{2,0} \\ \vdots \\ c_{0,2} \\ c_{0,1} \\ c_{0,0} \end{pmatrix}. \tag{6.17}$$

Operators can now be written as $(n+1)^2 \times (n+1)^2$ matrices.

Given a cascaded system, we can now compute the pure steady state $|\phi\rangle$ via the following steps:

1. Compute null(C), the null space of C. Let $U = \text{null}(C)$ be a k dimensional subspace defined by k orthonormal vectors $u_1, u_2, \ldots, u_k$.

2. Compute null(D), the null space of D. Let $V = \text{null}(D)$ be a l dimensional subspace defined by l orthonormal vectors $v_1, v_2, \ldots, v_l$. $|\phi\rangle$ must lie within both null(C) and null(D) from previous analysis.

3. Define a matrix W such that the columns of W are the vectors $u_1, \ldots, u_k, v_1, \ldots, v_l$.

4. Compute $D = [\text{Rank}(U) + \text{Rank}(V)] - \text{Rank}(W)$. If $D = 0$, then by the Invertible Matrix Theorem,[5] the columns of W are independent and thus U and V share no common subspace and there does not exist a pure steady state. If $D = m$, then U and V share a common subspace of dimension m, and there will exist m orthogonal pure steady states of the system.

5. Suppose $D \neq 0$, then our aim is to determine the states that span the common subspaces. To do this, compute T, the reduced echelon form of W. Consider the case when $D = 1$, and let p be such that the p^{th} column is a linear combination of columns

[5] A good discussion of the theorem can be found in stand linear algebra textbooks, such as Chapter 5 of Lay [43].

1 to $p-1$. Then T will be of the form

$$T = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & T_{1,p} & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & T_{2,p} & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 & T_{3,p} & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & & & 1 & T_{p-1,p} & 0 & & 0 \\ & & & & 0 & 0 & 1 & \dots & 0 \\ & & & & & & 0 & \ddots & \vdots \\ & & \mathbf{0} & & & & & \ddots & 1 \\ & & & & & & & & 0 \\ & & & & & & & & \vdots \\ & & & & & & & & 0 \end{bmatrix}. \tag{6.18}$$

As row reduction does not change the ratio between the individual columns, the reduced form clearly indicates that

$$w_p = \sum_{j=1}^{p-1} T_{j,p} w_j, \tag{6.19}$$

where w_j indicates the j^{th} column of W. Noting that $w_j = u_j$ for $j \leq k$ and $w_j = v_{j-k}$ for $j > k$, we have

$$w_p = v_{p-k} = \sum_{j=1}^{k} T_{j,p} u_j + \sum_{j=1}^{p-k-1} T_{j+k,p} v_j \tag{6.20}$$

$$\Rightarrow \qquad \sum_{j=1}^{k} T_{j,p} u_j = v_{p-k} - \sum_{j=1}^{p-k-1} T_{j+k,p} v_j. \tag{6.21}$$

Note that the left hand side of the above equation is a linear combination of vectors in null(C) and the right hand side in null(D), this vector belongs to both null(C) and null(D) and hence must be a steady state of the system. That is

$$|\phi\rangle = \sum_{j=1}^{k} T_{j,p} u_j. \tag{6.22}$$

can be regarded as the solution to the system.

The algorithm uses standard methods of linear algebra that are readily available in numerical packages such as Matlab. While it is numerically efficient, it is worth noting that finding the null space of an operator is a taxing process. Thus on standard desktop computers, our numerical method is limited to around $n = 15$. While this is certainly not enough to draw any conclusions about the behaviour of the steady state in the large-n limit, it does unveil a number of noteworthy features that provide gateways for more thorough investigation.

6.4.2 Special Case: Cascaded Qubits

The best way to verify an algorithm is to apply it to a system that we have prior knowledge of. Therefore, let us reconsider the Cascaded Qubit System from the previous chapter, in a more general setting. Recall that previously we limited ourselves to the case where the cavities are symmetric with respect to parameters β_r and β_s. What occurs in the more general case? In matrix representation, we have operators

$$R_1 = \begin{bmatrix} 0 & 0 & \beta_{s,1}^* & 0 \\ 0 & 0 & 0 & \beta_{s,1}^* \\ \beta_{r,1}^* & 0 & 0 & 0 \\ 0 & \beta_{r,1}^* & 0 & 0 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 0 & \beta_{s,2}^* & 0 & 0 \\ \beta_{r,2}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_{s,2}^* \\ 0 & 0 & \beta_{r,2}^* & 0 \end{bmatrix},$$

and so

$$C = \begin{bmatrix} 0 & -\beta_{s,2}^* & \beta_{s,1}^* & 0 \\ -\beta_{r,2}^* & 0 & 0 & \beta_{s,1}^* \\ \beta_{r,1}^* & 0 & 0 & -\beta_{s,2}^* \\ 0 & \beta_{r,1}^* & -\beta_{r,2}^* & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} \mathbf{0} & & & \beta_{s,1}^*\beta_{r,2} - \beta_{r,1}\beta_{s,2}^* \\ & \beta_{s,1}^*\beta_{s,2} - \beta_{r,1}\beta_{r,2}^* & & \\ & & \beta_{r,1}^*\beta_{r,2} - \beta_{s,1}\beta_{s,2}^* & \\ \beta_{r,1}^*\beta_{s,2} - \beta_{s,1}\beta_{r,2}^* & & & \mathbf{0} \end{bmatrix}.$$

Notice that for general parameters, C is of full rank and hence has a null space of dimension 0, signifying that there exists no pure steady state. Thus, a pure steady state for the general cascaded qubit system only occurs when the system exhibits certain kinds of symmetry. Specifically, we require that

$$\beta_{r,1}\beta_{s,1} = \beta_{r,2}\beta_{s,2}. \tag{6.23}$$

Using the above relation, we can calculate

$$\text{null}(C) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{\beta_{r,1}^*}{\beta_{r,2}^*} \\ \frac{\beta_{r,1}^*}{\beta_{s,2}^*} & 0 \end{bmatrix} = [u_1, u_2]. \tag{6.24}$$

Now consider a general matrix D with $D_{3,2} = -D_{2,3}^*$ and $D_{4,1} = -D_{1,4}^*$ as the only non-zero

elements. We can calculate the spectral decomposition of $D = \Gamma\Lambda\Gamma^\dagger$ to be

$$\Gamma = \begin{bmatrix} 0 & 0 & i\frac{D_{1,4}}{|D_{1,4}|} & -i\frac{D_{1,4}}{|D_{1,4}|} \\ -i\frac{D_{2,3}}{|D_{2,3}|} & i\frac{D_{2,3}}{|D_{2,3}|} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \tag{6.25}$$

$$\Lambda = \begin{bmatrix} i|D_{2,3}| & 0 & 0 & 0 \\ 0 & -i|D_{2,3}| & 0 & 0 \\ 0 & 0 & i|D_{1,4}| & 0 \\ 0 & 0 & 0 & -i|D_{1,4}| \end{bmatrix}, \tag{6.26}$$

where in our case

$$D_{1,4} = \beta^*_{s,1}\beta_{r,2} - \beta_{r,1}\beta^*_{s,2}, \tag{6.27}$$

$$D_{2,3} = \beta^*_{s,1}\beta_{s,2} - \beta_{r,1}\beta^*_{r,2}. \tag{6.28}$$

It is clear that for general values of $D_{1,4}$ and $D_{2,3}$, D is non-degenerate, and none of the resulting eigenspaces are completely contained in null(C), and hence again, no pure steady state exists. So, what conditions do guarantee the existence of such a state?

Let us restrict ourselves to the special case where $\beta_{r,1}$, $\beta_{r,2}$, $\beta_{s,1}$, $\beta_{s,2}$ are real. Then it is clear that in order for a steady state to occur, D must be degenerate. That is, either $D_{1,4} = 0$ or $D_{2,3} = 0$.

The Symmetric System:

Suppose that we let $D_{1,4} = 0$, then the two cavity operating parameters must be identical up to a phase factor. That is

$$\beta_{s,1} = \pm\beta_{s,2}, \beta_{r,1} = \pm\beta_{r,2}. \tag{6.29}$$

Hence, this corresponds to the condition of symmetric cavities that was analyzed extensively in the previous chapter. Applying this condition, D will have a degenerate eigenspace (with eigenvalue 0) of the form:

$$\Gamma_1 = \begin{bmatrix} i & -i \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}. \tag{6.30}$$

Comparing this to Null(C) in (6.24), we deduce that the equilibrium state for this system is

$$|\phi\rangle_{sym} = \frac{1}{\sqrt{|\beta_r|^2 + |\beta_s|^2}} \begin{bmatrix} \beta_s \\ 0 \\ 0 \\ \beta_r \end{bmatrix}. \tag{6.31}$$

This result is in agreement with previous analysis. In addition it shows that our restriction to symmetric systems was not arbitrary, but rather a condition that helped maximize our knowledge of the system at steady state, thereby maximizing the chances of generating entanglement.

Anti-symmetric System

In a completely analogous calculation, we can enforce that $D_{2,3} = 0$. Resulting in

$$\beta_{s,1} = \pm\beta_{r,2} \qquad \beta_{r,1} = \pm\beta_{s,2} \tag{6.32}$$

and the steady state is of the form

$$|\phi\rangle_{asym} = \frac{1}{\sqrt{|\beta_{r,1}|^2 + |\beta_{s,2}|^2}} \begin{bmatrix} 0 \\ \beta_{s,2} \\ \beta_{r,1} \\ 0 \end{bmatrix} \tag{6.33}$$

This system corresponds to the case where the rate of a transition from $|0\rangle$ to $|1\rangle$ in the first cavity is equal to the transition from $|1\rangle$ to $|0\rangle$ in the second. Hence we consider this to be the 'antisymmetric' system

6.4.3 Conditions for a Pure Steady State

While we have only demonstrated the conditions for a pure steady state in the case of $n = 2$, numerical results demonstrate that the conditions also hold for large systems.

Proposition 8 (Conditions for a Pure Steady State) *Given a cascaded cavity system where we restrict the transition rates to real numbers, only two conditions permit the existence of pure steady states.*

- **Symmetric Systems:** $\beta_{r,1} = \beta_{s,1}$, $\beta_{r,2} = \beta_{s,2}$
- **Antisymmetric Systems:** $\beta_{r,1} = \beta_{s,2}$, $\beta_{r,2} = \beta_{s,1}$

While we could analyze both conditions individually, we notice that they essentially describe the same system. Antisymmetric systems can be easily converted into symmetric systems be a relabelling of the states $|0\rangle$ and $|1\rangle$. Hence, we can characterize the entanglement of all pure steady states by only considering the case of a symmetric system. Needless to say, this saves us a lot of time and precious trees.

6.4.4 Numerical Solutions for Steady State

The development of a systematic numerical method to solve for steady states allows us to compute the steady state for a varied range of cavity systems. In this section, we list a representative selection of such solutions, and state some of the observed results and trends qualitatively. This information can then be used to ascertain general properties of the system.

We restrict ourselves to the case where the cavities are symmetric, as this was shown previously to be quite general. As in the case of qubits, we define $r = \frac{\beta_s}{\beta_r}$ to simplify our notation.

In addition, we introduce a matrix notation for pure states of the cascaded cavity system.

$$|\phi\rangle = \sum_{j,k=0}^{n} c_{j,k}|-n/2+j\rangle_1|-n/2+k\rangle_2 = \begin{bmatrix} c_{n,n} & c_{n-1,n} & \cdots & c_{0,n} \\ c_{n,n-1} & c_{n-1,n-1} & \cdots & c_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,0} & c_{n-1,0} & \cdots & c_{0,0} \end{bmatrix} \quad (6.34)$$

While it is typical to represent states as vectors, the above form is much more convenient for analysis of entanglement. Recall that the entanglement can be defined by the amount of uncertainty introduced about one of the cavity systems when one makes a projective measurement on the other cavity system.

Suppose now that Alice represents the observer for the first cavity system, and Bob the second. A measurement by Alice will collapse the state of the system into an ensemble of states $|m_i\rangle$ with probabilities $|m_i|^2$. Therefore, a quick glance at the matrix will shed light on the entanglement between the two systems. (See Figure 6.3).

For example, in the case of the qubit system in the limit of resonance, the steady state of the system is represented by

$$|\phi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (6.35)$$

A quick glance will immediately reveal that the two columns are orthogonal, and as they have equal norm, the state must exhibit maximal entanglement.

The steady state solutions for $n = 1, 2, 3$ are shown for the case where $r = 0.8$:

$$n = 1 \quad \begin{bmatrix} 0.6247 & 0 \\ 0 & 0.7809 \end{bmatrix} \quad (6.36)$$

$$n = 2 \quad \begin{bmatrix} 0.3902 & 0 & -0.4878 \\ 0 & 0 & 0 \\ -0.4878 & 0 & 0.6098 \end{bmatrix} \quad (6.37)$$

$$n = 3 \quad \begin{bmatrix} 0.3040 & 0 & -0.3291 & 0 \\ 0 & 0 & 0 & -0.4114 \\ -0.3291 & 0 & 0 & 0 \\ 0 & -0.4114 & 0 & 0.5938 \end{bmatrix}. \quad (6.38)$$

Consider the case where $n = 2$, a quick calculation will show that the first and third columns of the matrix represent the exact same state up to a constant factor. This implies that when Alice makes a measurement on her system, she will either get 0 or 2 atoms in the $|1\rangle$ state, and regardless of her measurement, Bob's cavity will always be in the same state, namely $\mathcal{N}[0.3902|1\rangle - 0.4878|0\rangle]$. Hence a projective measurement made by Alice introduces no uncertainty to Bob, and the state is not entangled at all.

For larger n, it is more useful to observe the results graphically. We can plot the probability amplitude of the individual basis states for each steady state $|\phi\rangle$. For example Figure 6.4

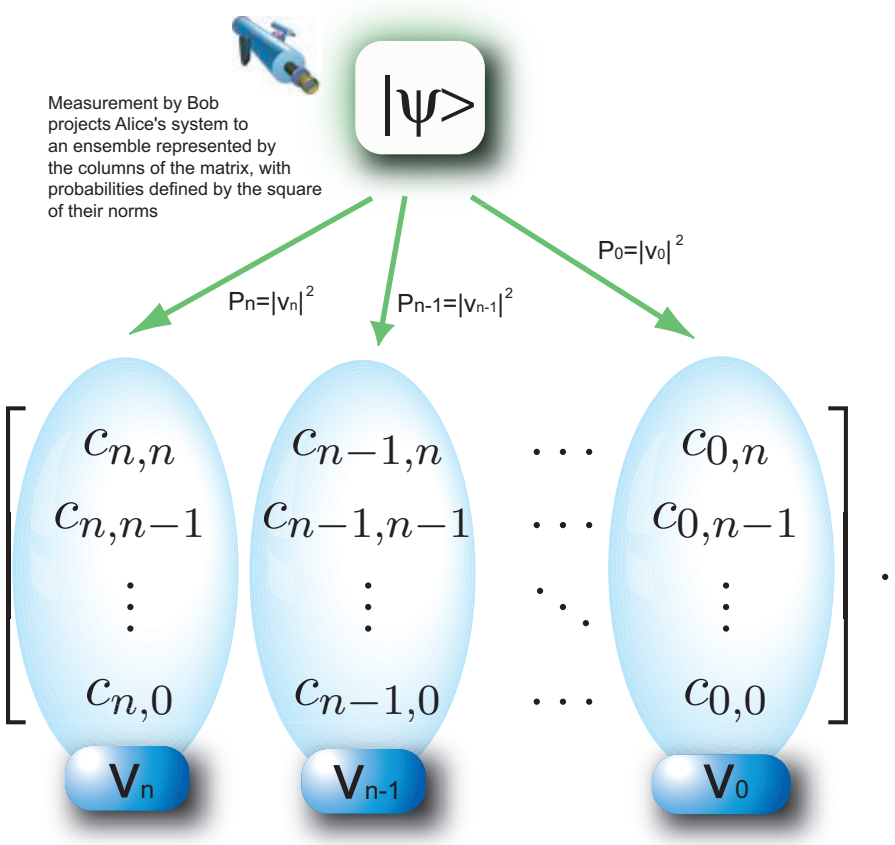

Figure 6.3: The natural unravelling of a state matrix. The columns of such a matrix represent an ensemble whose entropy can often be evaluated or at least estimated by sight. This entropy is then equal to the entanglement of the system.

demonstrates the steady state of the system in the limit of resonance for $n = 11$ and $n = 12$. It is apparent that the lack of entanglement is also true for the case where $n = 12$. In fact numerical simulations up to $n = 20$ show that this is a general feature for n even. That is, the chances of measuring an odd number of atoms in $|1\rangle$ for either cavity is always 0, and regardless of Alice's measurement result, the state of the second cavity is unaffected.

Now consider the case where $n = 11$; entanglement clearly exists due to the parity anti-correlations between the cavities. To be exact, suppose Alice made a projective measurement and got an even result (that is, an even number of atoms in the $|1\rangle$ state), then a careful look at the graph shows that the only nonzero probabilities for Bob are odd (that is, an odd number of atoms in the $|1\rangle$ state).[6] In contrast, if Alice were to have an odd result, Bob's measurement must be even. This fact alone tells us that the resulting ensemble after Alice's

[6]For the sake of brivity, from now on, we will call a state even (odd) if it has an even (odd) number of atoms in the $|1\rangle$ state.

measurement will include at least two sets of states that are orthogonal to each other. In particular, from the symmetry of resonance, the chances of Alice measuring odd or even are identical, and hence the entanglement at steady state will be at least 1, regardless of which odd value n takes. As the entanglement is caused by a parity anti-correlation, we will call it parity entanglement.

A second feature that is worth noting is that in the limit of resonance, the steady state is symmetric with respect to a relabelling of $|0\rangle$ and $|1\rangle$. That is, the number of atoms in states $|1\rangle$ and $|0\rangle$ coincide. However, if $r \neq 1$, then there is a bias towards the state that is the destination of the stronger of the two transitions. This is of course to be expected, and in the limit where β_r or β_s is zero, all atoms will accumulate in $|0\rangle$ or $|1\rangle$ respectively.

We can summarise the results observed from numerical simulations succinctly:a

- Steady states are completely symmetric with respect to cavity interchange.
- Steady states only exist when the cascaded system exhibits either symmetry or anti-symmetry.
- If $\beta_r > \beta_s$, then $c_{j,k} \to 0$ as $j, k \to n/2$. Conversely, if $\beta_r < \beta_s$, then $c_{j,k} \to 0$ as $j, k \to -n/2$, for large n.
- Let $J_\pm = J_1 \pm J_2$, i.e, the sum and difference angular momenta respectively. Then the only states with non-zero coefficients are the ones such that J_- is even and $J_+ = -n, -n+2, \ldots, n$.

Although numerical simulations were able to substantially improve our knowledge about the steady state behaviour, they certainly cannot give a complete picture. In particular, the numerics involve finding the null space of an $(n+1) \times (n+1)$ vector, and hence have approximate complexity $O(n^5)$.[7] While this is not exponential, it still grows too quickly for any standard desktop computer that is available to a poor MSc student. Due to this, the numerics cannot elucidate the properties of steady state for large n.

In particular, what happens to the entanglement? Does it grow indefinitely, or is it bounded? How does the parameter r affect the entanglement for large n? To answer these questions, and formally prove these conjectures inspired by our observations, we will need an analytical approach.

6.5 Analytical Solution for Steady State

Despite the complexity of the cascaded cavity system, it turns out that we can derive analytically many of the properties observed numerically. In fact, we will see that it is possible to solve for the steady state implicitly, in terms of a multivariate recursive relation.

[7] This comes from the fact that computing the null space of a matrix is approximately cubic with respect to the size of the matrix [44].

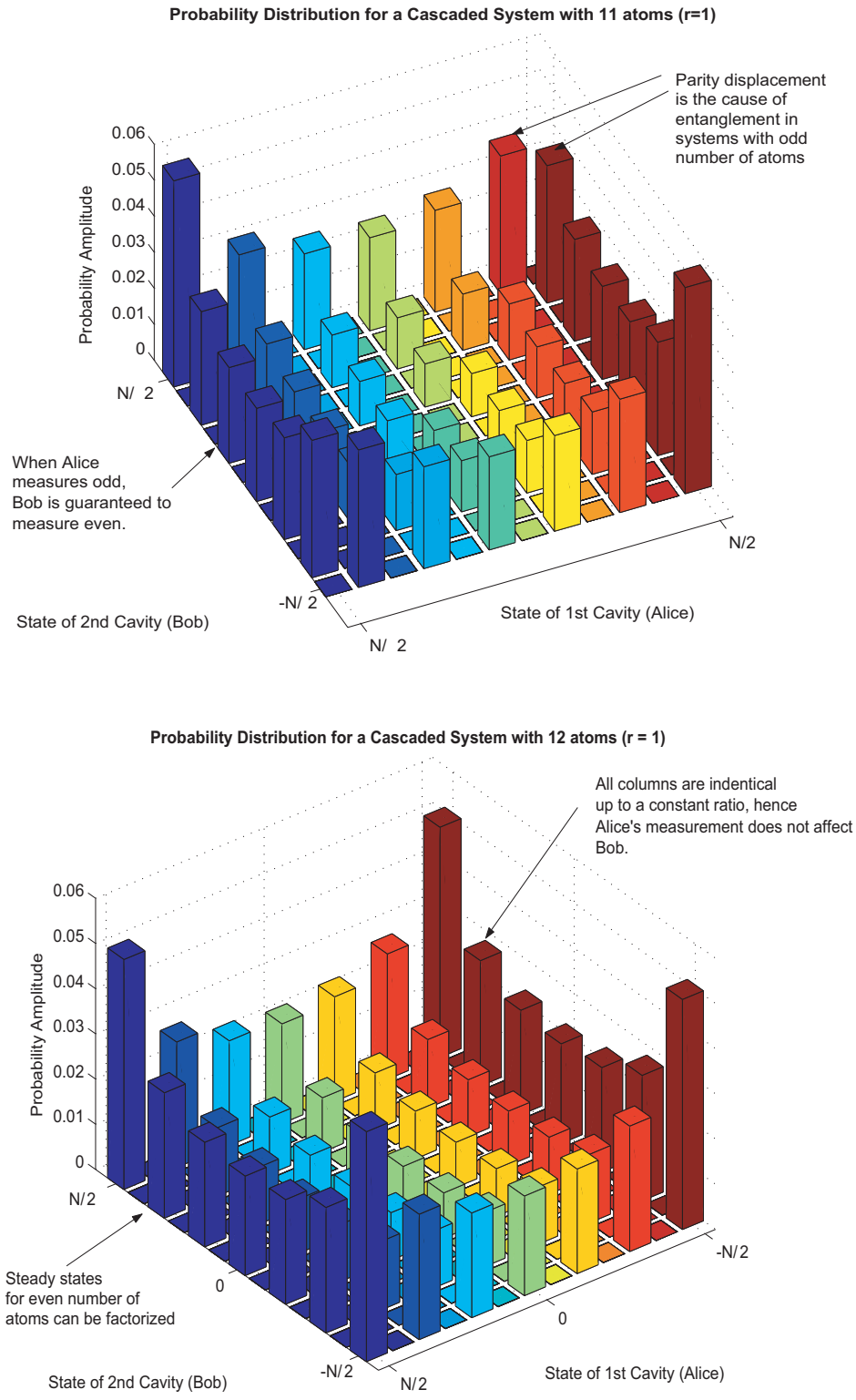

Figure 6.4: Probability amplitude of the steady state at resonance when $n = 11$ and $n = 12$. Notice that a slight parity shift causes a vast difference in the behavior of entanglement within the system.

6.5.1 Product Steady States

Recall that we have conjectured that the steady state of a cascaded cavity system with an even number of atoms in each cavity shows no entanglement. Can this be proven rigorously? Let us suppose $|\phi\rangle$ is the steady state of our system, and assume it can be written as

$$|\phi\rangle = |\phi_1\rangle|\phi_2\rangle, \tag{6.39}$$

where $|\phi_1\rangle$ and $|\phi_2\rangle$ are the states for the first and second cavities respectively. The product state assumption allows us to greatly simplify the conditions for equilibrium. We have

$$\begin{aligned} C|\phi\rangle &= (R_1|\phi_1\rangle)|\phi_2\rangle - |\phi_1\rangle(R_2|\phi_2\rangle) && (6.40)\\ &= \left[\beta_r^* S_1|\phi_1\rangle + \beta_s^* S_1^+|\phi_1\rangle\right]|\phi_2\rangle - |\phi_1\rangle\left[\beta_r^* S_2|\phi_2\rangle + \beta_s^* S_2^+|\phi_2\rangle\right] = 0. && (6.41)\end{aligned}$$

This results in symmetric conditions on $|\phi_1\rangle$ and $|\phi_2\rangle$:

$$\begin{aligned} \beta_r^* S_1|\phi_1\rangle + \beta_s^* S_1^\dagger|\phi_1\rangle &= c_1|\phi_1\rangle \\ \beta_r^* S_2|\phi_2\rangle + \beta_s^* S_2^\dagger|\phi_2\rangle &= c_1|\phi_2\rangle. \end{aligned} \tag{6.42}$$

for some constant c_1. Now from the condition $H_s|\phi\rangle = 0$, we have

$$(R_2^\dagger R_1 - R_1^\dagger R_2)|\phi_1\rangle|\phi_2\rangle = 0 \tag{6.43}$$

$$\Rightarrow \qquad \left(|\beta_r|^2 - |\beta_s|^2\right)\left(S_2^\dagger S_1 - S_1^\dagger S_2\right)|\phi_1\rangle|\phi_2\rangle = 0 \tag{6.44}$$

Note that this second condition is automatically satisfied for the case of resonance. Thus, at resonance, we have only one restriction for the pure steady state (6.42), which results in $n+1$ independent solutions. This is a generalization of the non-uniqueness of steady state given for the cascaded qubit system, which has two independent steady states.

Recall that the cascaded system will only settle to a well defined steady state when the system is non-resonant. Therefore, we will restrict our analysis to non-resonant systems. In this case, (6.43) implies

$$[R_1|\phi_1\rangle]\left[R_2^\dagger|\phi_2\rangle\right] = \left[R_1^\dagger|\phi_1\rangle\right][R_2|\phi_2\rangle] \tag{6.45}$$

which results in one further restriction on $|\phi_1\rangle$ and $|\phi_2\rangle$, i.e,

$$S_1|\phi_1\rangle = c_2 S_1^\dagger|\phi_1\rangle, \qquad S_2|\phi_2\rangle = c_2 S_2^\dagger|\phi_2\rangle. \tag{6.46}$$

If we substitute the above condition into (6.42), we obtain

$$\begin{aligned} (\beta_r c_2 + \beta_s) S_1^\dagger|\phi_1\rangle &= c_1|\phi_2\rangle, && (6.47)\\ (\beta_r c_2 + \beta_s) S_2^\dagger|\phi_2\rangle &= c_1|\phi_2\rangle. && (6.48)\end{aligned}$$

which is simply an eigenfunction equation for the operators $S_1^\dagger$ and $S_2^\dagger$. However, $S_1^\dagger$ and $S_2^\dagger$ are ladder raising operators for angular momentum, and we know that such operators have no non-zero eigenvalues. Thus, we can immediately set $c_1 = 0$. This further simplifies the steady state condition.

Proposition 9 (Steady State Condition for Separable States) *Suppose* $|\phi\rangle = |\phi_1\rangle|\phi_2\rangle$ *is a separable state of the cascaded system. Then it must satisfy the conditions*

$$R_i|\phi_i\rangle = \beta_s^* S_i|\phi_i\rangle + \beta_r^* S_i^\dagger|\phi_i\rangle = 0, \qquad i = 1, 2. \tag{6.49}$$

That is, each individual cavity must not be emitting any photons.

Proposition 9 allows for a direct analytical solution to our system. Since the conditions on $|\phi_1\rangle$ and $|\phi_2\rangle$ are identical, we simply need to consider the solution to the problem of a single cavity state $|\psi\rangle$ subject to

$$\beta_r^* S|\psi\rangle + \beta_s^* S^\dagger|\psi\rangle = 0. \tag{6.50}$$

where S and $S^\dagger$ are the angular momentum raising and lowering operators. We now expand $|\psi\rangle$ in the angular momentum eigenstates such that

$$|\psi\rangle = \sum_{m=-n/2}^{n/2} c_m|m\rangle. \tag{6.51}$$

Substituting into (6.50), we find that

$$\begin{aligned}
\beta_r^* \sum_{m=-n/2+1}^{n/2} c_m\sqrt{(\tfrac{n}{2}+m)(\tfrac{n}{2}-m+1)}|m-1\rangle &= -\beta_s^* \sum_{m=-n/2}^{n/2-1} \sqrt{(\tfrac{n}{2}-m)(\tfrac{n}{2}+m+1)}|m+1\rangle \\
\beta_r^* \sum_{m=-\frac{n}{2}}^{\frac{n}{2}-1} c_{m+1}\sqrt{(\tfrac{n}{2}+m+1)(\tfrac{n}{2}-m)}|m\rangle &= -\beta_s^* \sum_{m=-\frac{n}{2}+1}^{n/2} c_{m-1}\sqrt{(\tfrac{n}{2}-m+1)(\tfrac{n}{2}+m)}|m\rangle.
\end{aligned}$$

Using orthogonality of angular momentum states, we obtain a boundary value difference equation.

$$c_{m+1} = -\frac{\beta_s^*}{\beta_r^*}\sqrt{\frac{(\frac{n}{2}-m-1)(\frac{n}{2}+m)}{(\frac{n}{2}+m+1)(\frac{n}{2}-m)}}c_{m-1}, \qquad c_{-\frac{n}{2}+1} = c_{\frac{n}{2}-1} = 0, \qquad m = -\frac{n}{2}+1, \ldots, \frac{n}{2}-1. \tag{6.52}$$

While the recurrence is a little too complex to be solved explicitly, it elucidates a number of useful results

Proposition 10 (Separable Steady States for Even Atom Numbers) *The steady state of the system is separable if and only if the number of atoms in each individual cavity,* n*, is even. Furthermore, the steady state will be of the form*

$$|\phi\rangle = \sum_{j,k=-n/2}^{n/2} c_j c_k|j\rangle|k\rangle, \qquad \text{with } c_k = 0 \qquad \forall k = -\frac{n}{2}+1, -\frac{n}{2}+3, \ldots \frac{n}{2}-1. \tag{6.53}$$

If n *is odd, then a separable steady state does not exist.*

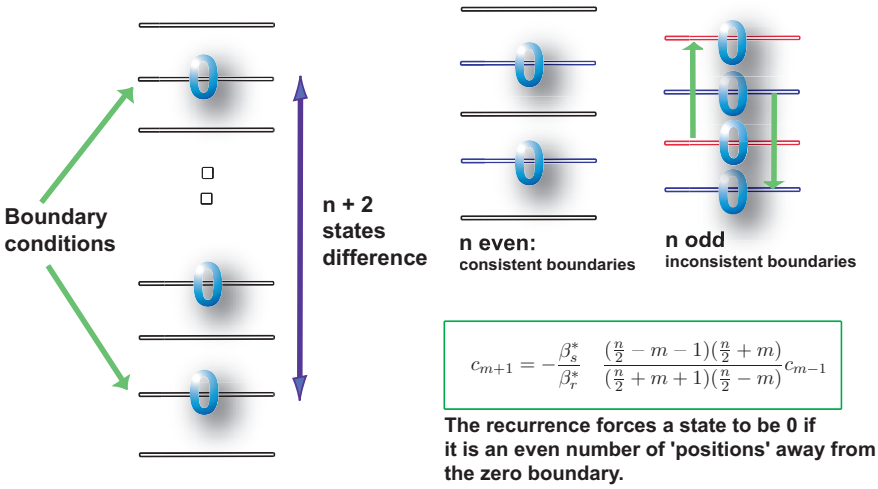

Figure 6.5: The nature of the recurrence relation implies that any state that is an even number of states away from the zero boundary condition is also zero. In the case of n even, the two boundaries are consistent, allowing non trivial solutions to exist. In the case of n odd, the boundaries are not consistent and only the trivial solution exists.

Proof:
We note that $(n/2-1)-(-n/2+1) = n+2$, which is even (odd) if n is even or odd. Since $c_m = 0$ implies that $c_{m+2k} = 0$ for all integer k, c_m is identically 0 if n is odd. The above result follows (Figure 6.5)

This formally proves one of the major conjectures made on the basis of numerical results, that product steady states can only be formed in systems with an even number of atoms in each cavity. Physically, this can be understood from the fact that for a steady state to be a product state, we require both cavities to emit no photons individually. To do so requires a superposition of the probabilities of a photon emission due to a $|0\rangle \rightarrow |1\rangle$ transition and a $|1\rangle \rightarrow |0\rangle$ transition, which can only be achieved when there is an odd number of states available (i.e. an even number of atoms).

6.5.2 Steady States of the General System

We now remove the restriction that the steady state be a product state to consider the most general case. The algebra becomes much more complicated and, again, no explicit representation of the solution can be obtained. However, the process allows us to prove more of the conjectures inspired by numerical results.

Qualitative Properties of the Steady State

We approach the problem by first considering the condition

$$\left[R_2^\dagger R_1 - R_1^\dagger R_2\right]|\phi\rangle = 0, \tag{6.54}$$

which, for the symmetric system, is equivalent to

$$\left[S_2^\dagger S_1 - S_1^\dagger S_2\right]|\phi\rangle = 0. \tag{6.55}$$

We now expand $|\phi\rangle$ in the product basis, such that

$$|\phi\rangle = \sum_{j,k=-n/2}^{n/2} c_{j,k}|j\rangle|k\rangle \tag{6.56}$$

whereupon (6.55) implies that

$$\begin{aligned}\sum_{j,k=-\frac{n}{2}}^{\frac{n}{2}} \left[(\frac{n}{2}+j)(\frac{n}{2}-j+1)(\frac{n}{2}-k)(\frac{n}{2}+k+1)\right]^{\frac{1}{2}} c_{j,k}|j-1\rangle|k+1\rangle = \\ \sum_{j,k=-\frac{n}{2}}^{\frac{n}{2}} \left[(\frac{n}{2}-j)(\frac{n}{2}+j+1)(\frac{n}{2}+k)(\frac{n}{2}-k+1)\right]^{\frac{1}{2}} c_{j,k}|j+1\rangle|k-1\rangle,\end{aligned} \tag{6.57}$$

which gives us the recursive relation

$$c_{j+1,k-1} = \left[\frac{(\frac{n}{2}-j+1)(\frac{n}{2}+j)(\frac{n}{2}+k+1)(\frac{n}{2}-k)}{(\frac{n}{2}+j+1)(\frac{n}{2}-j)(\frac{n}{2}-k+1)(\frac{n}{2}+k)}\right]^{\frac{1}{2}} c_{j-1,k+1}, \tag{6.58}$$

subject to the boundary conditions

$$c_{\frac{n}{2}+1,k} = c_{\frac{n}{2}-1,k} = c_{j,\frac{n}{2}-1} = c_{j,\frac{n}{2}+1} = 0, \qquad j,k = -\frac{n}{2}+1,\ldots,\frac{n}{2}-1. \tag{6.59}$$

While this relation does not uniquely define the equilibrium solution, it does allow us to state a number of nice properties. In particular, the nature of the term dependence allows us to employ mathematical techniques analogous to the product steady states derived earlier to set many of the coefficients to zero.

Proposition 11 (Zero Coefficients of Steady State) *Let $|\phi\rangle$ be the steady state of the cascaded cavity system. Then, we can write*

$$|\phi\rangle = \sum_{j,k=-n/2}^{n/2} c_{j,k}|j\rangle|k\rangle. \tag{6.60}$$

For n even, we have the additional condition

$$c_{j,k} \neq 0 \qquad \text{only if } j,k = -\frac{n}{2}, \frac{n}{2}+2, \ldots \frac{n}{2}. \tag{6.61}$$

Alternatively for n odd

$$c_{j,k} \neq 0 \quad \text{only if } j,k = -\frac{n}{2}+1, -\frac{n}{2}+3, \ldots \frac{n}{2} \text{ for } j+k>0$$
$$\text{and } j,k = -\frac{n}{2}, -\frac{n}{2}+2, \ldots \frac{n}{2}-1 \text{ for } j+k<0. \tag{6.62}$$

Proof: The idea behind the proof is that the recurrence relation (6.58) can be separated into $2n+1$ individual recurrences, relating the coefficients $c_{j,k}$ such that $j+k=m$, with $m=-n,-n+1,\ldots,n$. Each of these individual recurrences has the same algebraic dependence as each of the individual cavity recurrences given for the product steady states. We can then apply the same methods to say that any of these diagonal lines with an even number of states will be identically zero, and every second state of any recurrence that relates an odd number of terms will also be zero.

Explicitly, denote the set of coefficients that satisfy $j+k=m$ by $\mathcal{C}_m$. Then $\mathcal{C}_m$ has $n-|m|+1$ elements, such that (see Figure 6.6):

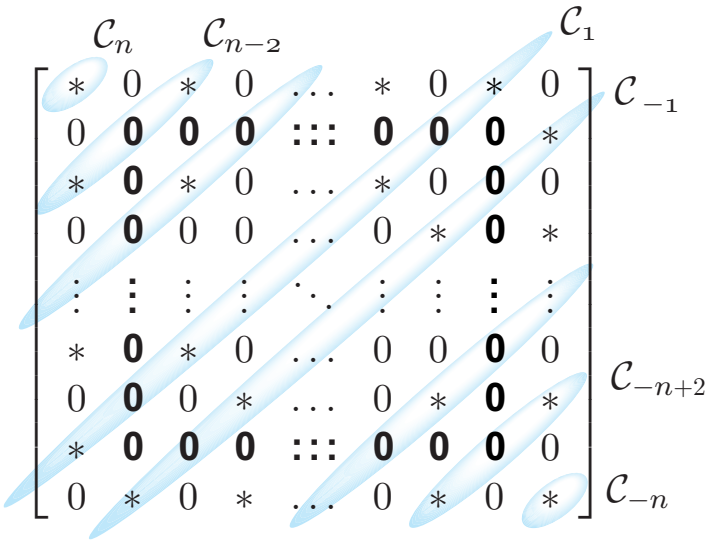

Figure 6.6: The coefficients of $\mathcal{C}_m$ are highlighted for each odd m. The numbers in bold correspond to the explicit boundary conditions. Note that only the highlighted recurrences contain an odd number of states, allowing consistency with the boundary conditions (this figure assumes that n is odd). In this figure, we let $*$ denote an arbitrary, non-zero number.

$$\mathcal{C}_m = \left\{ c_{-\frac{n}{2},\frac{n}{2}-|m|}, c_{-\frac{n}{2}+1,\frac{n}{2}-|m|-1}, c_{\frac{n}{2}-|m|,-\frac{n}{2}}, \right\}, \qquad m<0. \tag{6.63}$$
$$\mathcal{C}_m = \left\{ c_{-\frac{n}{2}+|m|,\frac{n}{2}}, c_{-\frac{n}{2}+|m|+1,\frac{n}{2}-1}, c_{\frac{n}{2},-\frac{n}{2}+|m|}, \right\} \qquad m \geq 0. \tag{6.64}$$

If we make the substitution $k = m - j$ in each of the sets of coefficients $\mathcal{C}_m$ and the recurrence (6.58), we arrive at $2n + 1$ univariate recurrences of the form

$$c_{j+1,m} = \left[\frac{(\frac{n}{2} - j + 1)(\frac{n}{2} + j)(\frac{n}{2} + m - j + 1)(\frac{n}{2} - m + j)}{(\frac{n}{2} + j + 1)(\frac{n}{2} - j)(\frac{n}{2} - m + j + 1)(\frac{n}{2} + m - j)}\right]^{\frac{1}{2}} c_{j-1,m}, \tag{6.65}$$

subject to the boundary conditions

$$c_{-\frac{n}{2}+1,m} = c_{\frac{n}{2}-|m|-1,m} = 0 \qquad m < 0, \tag{6.66}$$

$$c_{-\frac{n}{2}+|m|+1,m} = c_{\frac{n}{2}-1,m} = 0 \qquad m \geq 0. \tag{6.67}$$

We now note that as $c_j = 0 \Rightarrow c_{j\pm 2} = 0$ from (6.65), the recursive relation has a non-trivial solution only if the two zero boundary conditions are an even number of coefficients apart. That is when $n - |m| - 2$ is even.

For the case where n is odd, it follows that non-trivial solutions can only occur when m is odd. That is

$$c_{j,k} \neq 0 \qquad \text{only if } m = j + k \text{ is odd.} \tag{6.68}$$

It also follows that

$$c_{-\frac{n}{2}+1,m} = c_{-\frac{n}{2}+3,m} = \ldots = c_{\frac{n}{2}-|m|-1,m} = 0, \qquad m < 0. \tag{6.69}$$

$$c_{-\frac{n}{2}+|m|+1,m} = c_{-\frac{n}{2}+|m|+3,m} = \ldots = c_{\frac{n}{2}-1,m} = 0, \qquad m \geq 0. \tag{6.70}$$

Thus, we obtain Proposition 11 for n odd. The case for n is even can be treated identically, and has already been demonstrated when we assumed the solution was separable. ■

The above recurrence is just one of the two conditions which wholly specify the steady state. Recall that for $|\phi\rangle$ to be in equilibrium, it must also satisfy

$$[R_1 - R_2]\,|\phi\rangle = 0, \tag{6.71}$$

which implies

$$\beta_r^* \,[S_1 - S_2]\,|\phi\rangle = \beta_s^* \left[-S_1^\dagger + S_2^\dagger\right]|\phi\rangle, \tag{6.72}$$

and thus

$$\begin{aligned}
&\beta_r^* \sum_{j,k} c_{j,k} \left[(\frac{n}{2} + j)(\frac{n}{2} - j + 1)\right]^{\frac{1}{2}} |j-1\rangle|k\rangle \\
&- \beta_r^* \sum_{j,k} c_{j,k} \left[(\frac{n}{2} + k)(\frac{n}{2} - k + 1)\right]^{\frac{1}{2}} |j\rangle|k-1\rangle \\
&= \beta_s^* - \sum_{j,k} c_{j,k} \left[(\frac{n}{2} - j)(\frac{n}{2} + j + 1)\right]^{\frac{1}{2}} |j+1\rangle|k\rangle \\
&+ \beta_s^* \sum_{j,k} c_{j,k} \left[(\frac{n}{2} - k)(\frac{n}{2} + k + 1)\right]^{\frac{1}{2}} |j\rangle|k+1\rangle.
\end{aligned} \tag{6.73}$$

Making a change of indices, taking the inner product with respect to $|j\rangle|k\rangle$, and using orthogonality, we obtain a second recurrence for $c_{j,k}$.

$$\beta_r^* \left\{ \left[(\frac{n}{2}+j+1)(\frac{n}{2}-j) \right]^{\frac{1}{2}} c_{j+1,k} - \left[(\frac{n}{2}+k+1)(\frac{n}{2}-k) \right]^{\frac{1}{2}} c_{j,k+1} \right\} =$$
$$\beta_s^* \left\{ -\left[(n-j+1)(n+j) \right]^{\frac{1}{2}} c_{j-1,k} + \left[(\frac{n}{2}-k+1)(\frac{n}{2}+k) \right]^{\frac{1}{2}} c_{j,k-1} \right\} \tag{6.74}$$

To make sense of this relation, we can interpret it geometrically. Consider a rectangular lattice that specifies the values of $c_{j,k}$ at each coordinate (j,k). For a fixed j and k, the recurrence above relates the values of c for the four co-ordinates adjacent to (j,k).

For the case where n is odd, we already know from Proposition 11 that the solution will be of the form shown in Figure 6.7. That is, for each choice of the indices j, k, at least two of the above terms vanish due to boundary conditions. This allows application of the recurrence (6.74) into to several distinct cases shown

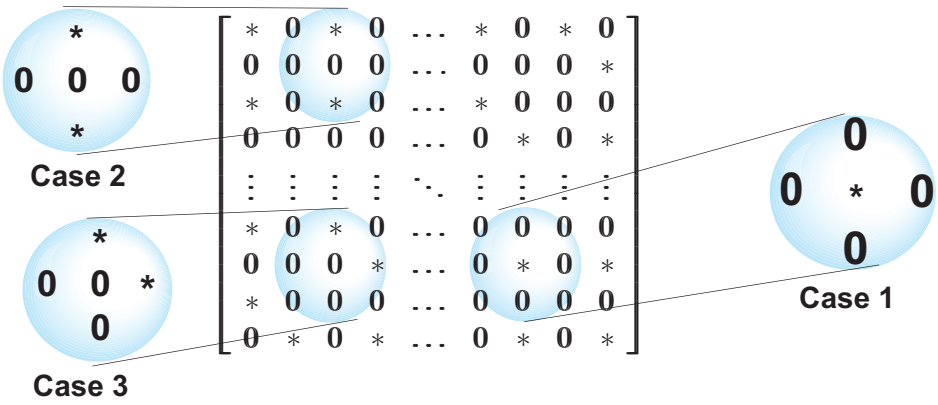

Figure 6.7: The three cases that (6.74) is reduced to, depending on the choice of (j,k), each lead to a separate relation, as certain coefficients relative to (j,k) are identically zero due to boundary conditions.

- **Case 1:** $j+k$ **is odd:** Trivial case. Here $c_{j+1,k} = c_{j-1,k} = c_{j,k-1} = c_{j,k+1} = 0$ and the recursion is automatically satisfied.
- **Case 2:** $m = j+k \neq 0$ **where** m **is even:** It is easy to see from Figure (6.7) that either $c_{j+1,k} = c_{j-1,k} = 0$ or $c_{j,k-1} = c_{j,k+1} = 0$. So (6.74) is reduced to

$$c_{j,k+1} = -\frac{\beta_s^*}{\beta_r^*} \left[\frac{(\frac{n}{2}-k+1)(\frac{n}{2}+k)}{(\frac{n}{2}+k+1)(\frac{n}{2}-k)} \right]^{\frac{1}{2}} c_{j,k-1},$$
$$c_{j+1,k} = -\frac{\beta_s^*}{\beta_r^*} \left[\frac{(\frac{n}{2}-j+1)(\frac{n}{2}+j)}{(\frac{n}{2}+j+1)(\frac{n}{2}-j)} \right]^{\frac{1}{2}} c_{j-1,k}. \tag{6.75}$$

We note that if we take the composition of the two equations above, the resulting relation is completely identical to that of (6.58). That is, the relations between the

coefficients $c_{j,k}$ that satisfy $j+k \neq 0$ is actually identical to the relations between coefficients in a cascaded even atom-number system.

- **Case 3:** $m = j+k = 0$**:** In this case, it is easy to see that $c_{j-1,k} = c_{j,k+1} = 0$ or $c_{j+1,k} = c_{j,k-1} = 0$. Thus, (6.74) becomes

$$c_{j,k+1} = \frac{\beta_s^*}{\beta_r^*}\left[\frac{(\frac{n}{2}-j+1)(\frac{n}{2}+j)}{(\frac{n}{2}+k+1)(\frac{n}{2}-k)}\right]^{\frac{1}{2}} c_{j-1,k},$$
$$c_{j+1,k} = \frac{\beta_s^*}{\beta_r^*}\left[\frac{(\frac{n}{2}-k+1)(\frac{n}{2}+k)}{(\frac{n}{2}+j+1)(\frac{n}{2}-j)}\right]^{\frac{1}{2}} c_{j,k-1}. \tag{6.76}$$

However, it is a simple matter to note that $j = -k$. Substituting into the above, we arrive at the simple relation

$$c_{j,k+1} = \frac{\beta_s^*}{\beta_r^*}c_{j-1,k} \qquad c_{j+1,k} = \frac{\beta_s^*}{\beta_r^*}c_{j,k-1} \qquad \text{when } j+k=0. \tag{6.77}$$

If we make a quick change of co-ordinates, the above statements are both equivalent to

$$c_{j+1,k+1} = \frac{\beta_s^*}{\beta_r^*}c_{j,k} \qquad \text{when } j+k=-1. \tag{6.78}$$

By combining (6.76) and (6.77), we arrive at a general implicit solution of the cascaded cavity system.

Proposition 12 (General Solution for the Symmetric Cascaded Cavity System) *Let ρ be the steady state of the symmetric cascaded cavity system with parameters β_r and β_s with n atoms in each individual cavity. Then $\rho = |\phi\rangle\langle\phi|$, with $|\phi\rangle = \sum_{j,k=-\frac{n}{2}}^{\frac{n}{2}} c_{j,k}|j\rangle|k\rangle$, is pure.*

Define the ratio function $f(n,m)$ by

$$f(n,m) = \left[\frac{(\frac{n}{2}-m+1)(\frac{n}{2}+m)}{(\frac{n}{2}+m+1)(\frac{n}{2}-m)}\right]^{\frac{1}{2}}. \tag{6.79}$$

***If** n **is odd**, $c_{j,k}$ satisfy the multivariate recursion*

$$c_{j,k+1} = -\frac{\beta_s^*}{\beta_r^*}f(n,k)c_{j,k}, \qquad c_{j+1,k} = -\frac{\beta_s^*}{\beta_r^*}f(n,k)c_{j,k}, \qquad j+k \neq 0 \tag{6.80}$$
$$c_{j+1,k+1} = \frac{\beta_s^*}{\beta_r^*}c_{j,k}, \qquad j+k=-1, \tag{6.81}$$

together with the boundary conditions

$$c_{\frac{n}{2}+1,k} = c_{\frac{n}{2}-1,k} = c_{j,\frac{n}{2}-1} = c_{j,\frac{n}{2}+1} = c_{j,-j} = 0. \tag{6.82}$$

***If** n **is even**, then $c_{j,k}$ satisfy*

$$c_{j,k+1} = -\frac{\beta_s^*}{\beta_r^*}f(n,k)c_{j,k-1}, \qquad c_{j+1,k} = -\frac{\beta_s^*}{\beta_r^*}f(n,j)c_{j-1,k}, \tag{6.83}$$

together with the boundary conditions

$$c_{\frac{n}{2}+1,k} = c_{\frac{n}{2}-1,k} = c_{j,\frac{n}{2}-1} = c_{j,\frac{n}{2}+1} = 0. \tag{6.84}$$

Together the above equations uniquely define the steady state for an arbitrary n

While ths recurrence relation is intractable when one tries to solve it for a general n, it presents an explicit algorithm for determining each of the coefficients that makes up the solution. The time taken to evaluate these coefficients grows only quadratically with respect to the atom number. Therefore the recurrence relations allow us to solve for steady states orders of magnitude larger than a direct numerical method would allow.

6.6 Entanglement Analysis at Steady State

With powerful analytical techniques available, we are finally in a position to achieve our goal, characterizing the amount of entanglement generated within a cascaded cavity system under steady state conditions for various different parameters.

We will consider two measures of entanglement, the absolute entanglement in terms of the Von Neumann measure, and the entanglement efficiency, both of which were defined in Chapter 2. The choice of Von Neumann entanglement is obvious, for it tells us the absolute quantity of entanglement that this system is capable of creating. The resource generated can then be converted to an equal number of Bell states for various quantum communication and information protocols.

In addition, however, we would like a measure of how good the system is at creating entanglement. That is, given two systems with d dimensions, could we entanglement them to a point where the two systems cannot be more entangled? For example, it would instinctively seem more impressive to generate an entanglement of 1 between qubits, than it would be to create the same amount of entanglement between two systems with 2^5 dimensions each. Noting that in this system, the number of states per subsystem is $d + 1$, we have

$$\mathrm{E}_n(|\phi\rangle) = -\frac{\mathrm{E}(\rho)}{\log_2(n+1)}, \tag{6.85}$$

where n is the number of atoms in each cavity. Thus, a state with $\mathrm{E}_n(\rho) = 1$ implies that the state features the maximum entanglement possible.

Finally, we will restrict all our results to the case where n is odd, for we have already shown that no entanglement exists when n is even.

6.6.1 Entanglement Efficiency at Steady State

Let us first consider how the parameter r affects the entanglement at steady state. From previous experience with the qubit system, we would expect maximal entanglement to occur at the point of resonance. This is demonstrated in Figure 6.8.

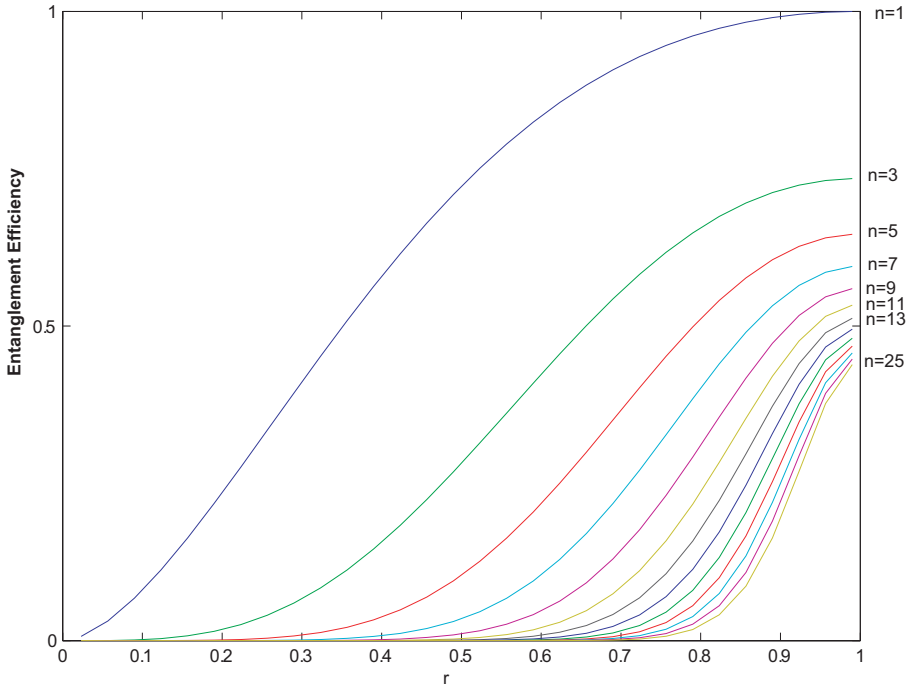

Figure 6.8: Entanglement efficiency versus r with various fixed n.

Note that we have chosen to omit the cases where $r > 1$, since the results are symmetric about $r = 1$[8]. Clearly the results demonstrate that maximum entanglement remains at the point of resonance.

Also of interest is that as we increase the number of atoms, the entanglement efficiency decreases regardless of β_s/β_r. In addition, the peak becomes narrower, such that any deviation from resonance results in a much more significant reduction in entanglement.

Figure 6.9 demonstrates this phenomenon with greater clarity, showing a steady monotonic decrease. A complete picture of how resulting entanglement depends on the parameters of the system is displayed in Figure 6.10.

We conclude that the steady state entanglement of the cascaded system is highly dependent on the number of atoms in each cavity. As the size of our system increases, the efficiency with which we can generate entanglement decreases. In addition, the system becomes much more sensitive to deviations from resonance, such that for large n, any small difference between β_r and β_s could destroy all entanglement within the system at steady state.

Physically we can understand this reduction of efficiency by considering the nature of the entanglement generated. Recall that the primary cause of the entanglement is the parity anti-correlations between the two atomic systems. However, such anti-correlations essentially

[8]If this is not obvious, consider a relabelling of $|0\rangle$ and $|1\rangle$, whereby β_r and β_s will also be interchanged.

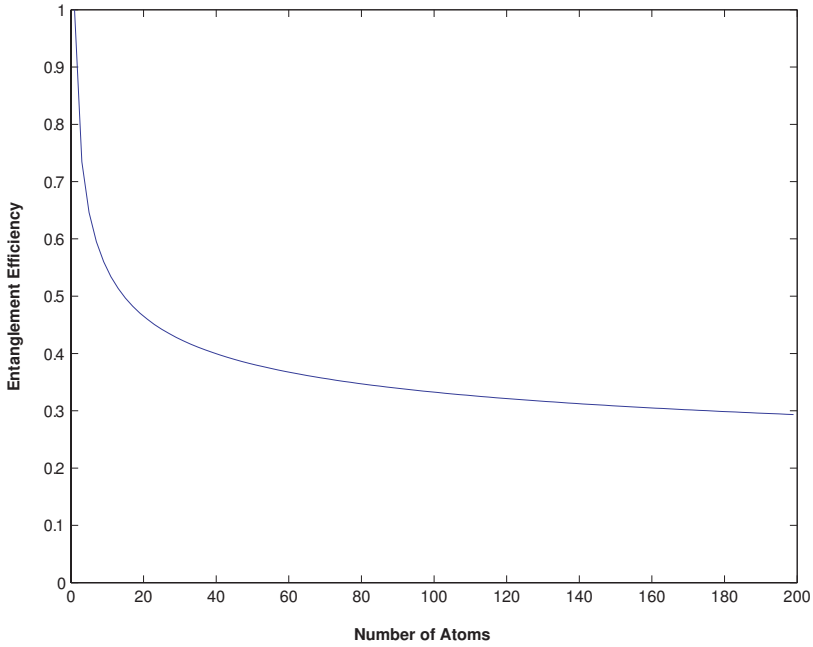

Figure 6.9: Plot of the entanglement efficiency for different n in the case where the cascaded system is resonant. We note that the entanglement decreases at a roughly logarithmic rate. We have omitted even values of n, which naturally have zero entanglement.

split the states of the system into two subsets, odd states and even states. The number of subsets do not scale with the dimensions of the system, and hence any entanglement caused primarily between correlations of those subsets will play a less prominent role with large numbers of atoms.

Absolute Entanglement at Steady State

The question however still remains, do we gain extra entanglement by employing extra atoms. Even if the efficiency were to decrease, a gain in entanglement would still signify that employing the extra atoms could be worthwhile. It may take 100 states to store the amount of entanglement that could be potentially stored with 10, but that would still be a considerable improvement from a qubit which can store at most one Bell state.

Since the absolute entanglement deviates from the normalized entanglement by a factor that depends only on n, it features the same dependence on β_r and β_s as before. Let us again consider the resonant system in Figure 6.11. In contrast to normalized entanglement, the absolute entanglement does increase with the number of atoms, albeit very slowly.

This implies that if perfect resonance can be achieved, it is possible to produce more entanglement by the addition of extra atoms within the individual cavities. However, one could

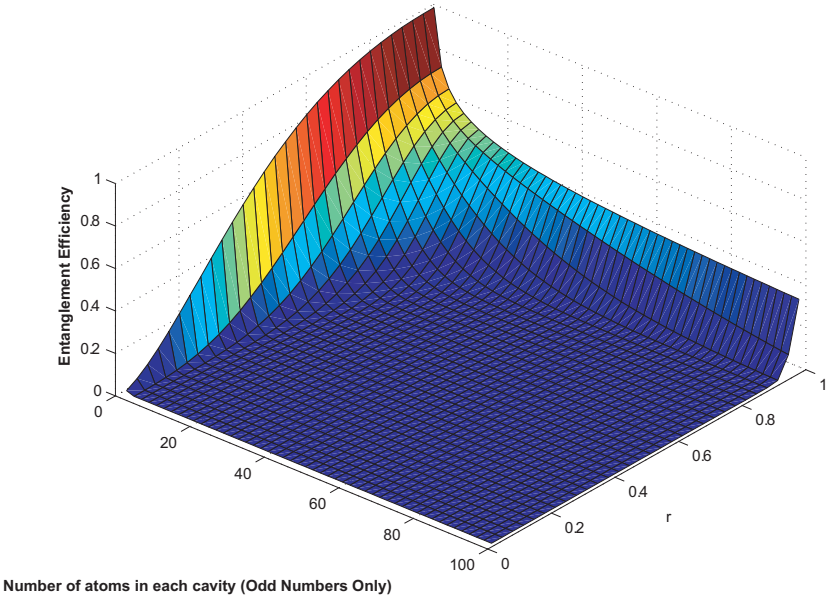

Figure 6.10: Entanglement efficiency for different r and n. We note that the graph has omitted the cases where n is even, as this leads to the trivial case where the resulting steady state has no entanglement.

ask about the feasibility of such an act, for recall that the closer the system is to resonance, the slower it evolves.

If we take into account non-resonant systems (see Figure 6.12), one can see that any deviation from resonance will drastically alter the asymptotic behaviour. Provided $r \neq 1$, the entanglement will peak for a certain value of $n = n_0$, and then tend to 0 as n increases further.

The numerical results show that the closer the system is to resonance, the greater the maximum entanglement, and the greater number of atoms we can place within the system before entanglement is destroyed.

Suppose quantum computation is limited by the difficulty of coupling large numbers of separate systems, we can consider the idea of using a cascaded cavity system to store more information with each pair of systems than is conventionally possible with two qubits. However, the upper limit in the amount of entanglement we can create will be bounded by how closely we can set the system to resonance.

6.7 The Many Atom Limit

A combination of analytical and numerical results have given us an almost complete understanding of how the steady state of the system will behave, save for one small gap. While

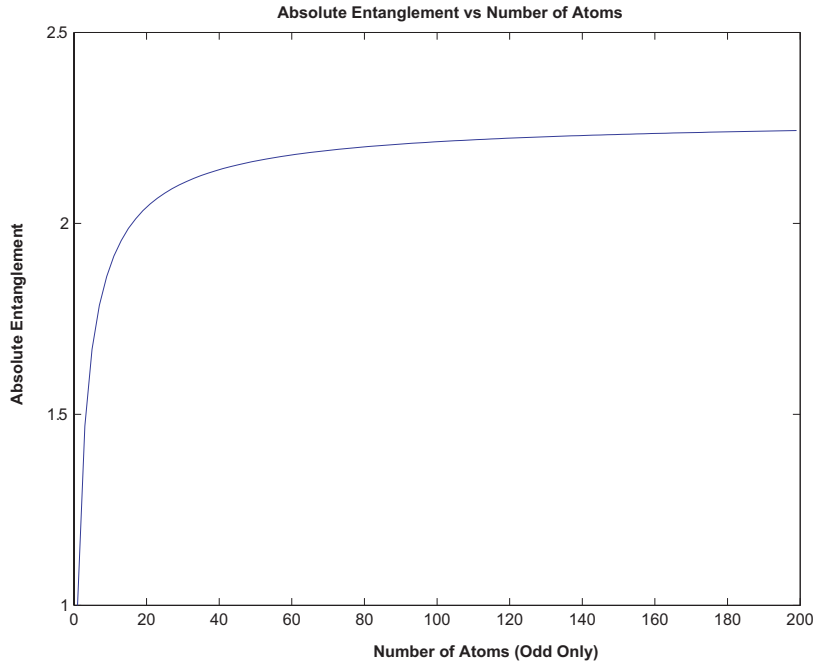

Figure 6.11: The absolute entanglement of a resonant system is a monotonely increasing function of n (for n odd).

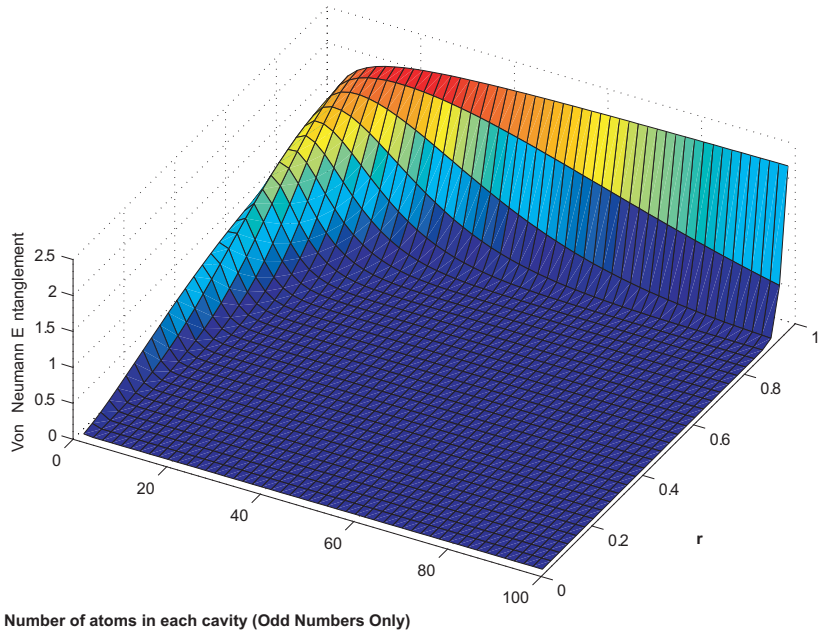

Figure 6.12: The absolute entanglement of the cascaded system for different values of n and r.

numerics have indicated that the entanglement of the system approaches zero in the limit as $n \to \infty$ unless it is at exact resonance, we have no formal proof that this is indeed the case. In particular, we would like to know, if a steady state of the system is separable in the limit of an infinite number of atoms, what is that steady state?

It turns out that evaluating this limit does not require taking the limit of our current results. In fact, the method that we use is quite elucidating, in that it highlights the connection between a discrete system with many states and a continuous system.

6.7.1 The Schwinger Approximation

Consider the steady state of the system as described by the analytical solution, one of the key features is that $|c_{j,k+2}| \propto |rc_{j,k}|$ and $|c_{j+2,k}| \propto |rc_{j,k}|$. This implies that if $|r| \leq 1$, then the probability of the system existing in a state such that most of the atoms in either cavity are in $|1\rangle$ is very small.

Let us assume that $|r| < 1$ without loss of generality. [9] Then in the limit as the number of atoms becomes very large, we can assume that the probability that the system is found in state such there is a significant fraction of atoms in $|1\rangle$ state is negligible. More precisely, we can assume that $n_{|1\rangle} \ll n$ for either cavity, where $n_{|1\rangle}$ denotes the number of atoms in state 1. While this assumption is an approximation for finite n, it becomes exact is the limit that n tends to infinity (for $|r| < 1$).

This limiting case can be best described in the Schwinger representation. As $n_{|1\rangle} \ll n$, the we can assume that the number of atoms in $|0\rangle$ is approximately n, that is $b^\dagger b \approx n$. Thus, $b^\dagger \approx b \approx \sqrt{n}$. This gives us a formal result for the system representation in the case of very large n. Physically, the approximation basically states that there so many atoms in state $|0\rangle$ that we can regard it as an endless sea of atoms (Figure 6.13).

Definition 12 (The Schwinger Approximation) *Consider a cascaded cavity system in the limit where $n \to \infty$. Then, we can rewrite the system via the representation*

$$S_i = \sqrt{n} a_i \qquad i = 1, 2 \tag{6.86}$$

$$J_{z,i} = -n \tag{6.87}$$

where a_i is the annihilation operator of the $|1\rangle$ mode in cavity i. That is, each atomic system is now algebraically identical to a bosonic mode.

Applying the approximation, the cascaded system is effectively reduced to a system of coupled oscillators. With

$$R_1 = \sqrt{n}\left(\beta_r^* a_1 + \beta_s^* a_1^\dagger\right), \tag{6.88}$$

$$R_2 = \sqrt{n}\left(\beta_r^* a_2 + \beta_s^* a_2^\dagger\right), \tag{6.89}$$

[9] Recall the states $|0\rangle$ and $|1\rangle$ can be relabelled.

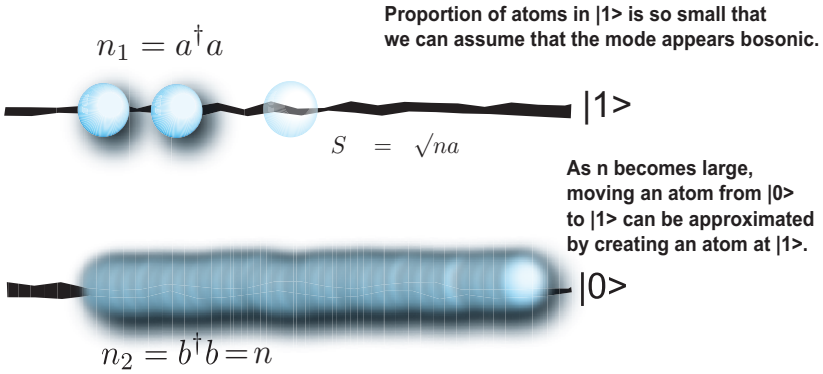

Figure 6.13: For large atom number cavities where all atoms begin in the $|0\rangle$ state. We can approximate the $|0\rangle$ mode as one that contains an infinite supply of atoms. That is, the annihilation of individual atoms in this mode can be neglected.

We can rewrite the dynamical system as

$$\dot{\rho} = i\left[\rho, H_{eff}\right] + C_1 \rho C_1^\dagger, \tag{6.90}$$

with

$$H_{eff} = in\left(|\beta_r|^2 - |\beta_s|^2\right)\left(a_2^\dagger a_1 - a_1^\dagger a_2\right), \tag{6.91}$$

$$C_1 = \sqrt{2}(R_2 - R_1). \tag{6.92}$$

Bosonic operators are much simpler to work with than angular momentum operators, due to their simple commutation relation. This simplification allows us to deduce an analytic solution to steady state in a much less taxing fashion than the general case.

6.7.2 The Limiting Steady State

With a complete reformulation of the cascaded cavity system in terms of bosonic operators, we are now in a position to calculate the limiting steady state and show that it exhibits no entanglement.

Recall that at steady state, we require the conditions $H_{eff}|\phi\rangle = 0$ and $C_1|\phi\rangle = 0$. This implies

$$(a_2^\dagger a_1 - a_1^\dagger a_2)|\phi\rangle = 0 \tag{6.93}$$

$$\left[\beta_r^*(a_1 - a_2) + \beta_s^*(a_1^\dagger - a_2^\dagger)\right]|\phi\rangle = 0 \tag{6.94}$$

We can simplify the system further by introducing the difference and sum operators,

$$A = \frac{1}{\sqrt{2}}(a_1 - a_2), \qquad B = \frac{1}{\sqrt{2}}(a_1 + a_2), \tag{6.95}$$

whereby

$$a_1 = \frac{1}{\sqrt{2}}(B + A), \qquad a_1 = \frac{1}{\sqrt{2}}(B - A). \tag{6.96}$$

We note that the operators A and B share identical commutation relations with a_1 and a_2.

$$[A, A^\dagger] = 1, \qquad [B, B^\dagger] = 1, \qquad [A, B] = 0. \tag{6.97}$$

Rewriting the steady state conditions (6.93) and (6.94) in terms of A and B, we have

$$\left(B^\dagger A - A^\dagger B\right)|\phi\rangle = 0 \tag{6.98}$$

$$\left(\beta_r^* A + \beta_s^* A^\dagger\right)|\phi\rangle = 0 \tag{6.99}$$

These conditions now form a system of equations that can be solved analytically.

The Condition $\left(\beta_r^* A + \beta_s^* A^\dagger\right)|\phi\rangle = 0$

: We first observe that any linear combination of an annihilation and a creation operator can be interpreted as a squeezing operator. Consider an operator $\hat{Q}$ such that

$$\hat{Q} = c_1 A + c_2 A^\dagger, \tag{6.100}$$

where $c_i = r_i e^{i\phi_i}$ and $r_1^2 - r_2^2 = 1$ so that

$$\hat{Q} = r_1 e^{i\phi_1} A + r_2 e^{i\phi_2} A^\dagger = \left(r_1 A + r_2 e^{-i(\phi_1 - \phi_2)} A^\dagger\right) e^{i\phi} \tag{6.101}$$

Noting from [51] that for $\xi = re^{i\phi}$

$$S_A^\dagger(\xi) A S_A(\xi) = \cosh(r) A + e^{-2i\phi} \sinh(r) A^\dagger \tag{6.102}$$

and comparing (6.102) with (6.101), it is clear that

$$r = \cosh^{-1}(r_1) \qquad \phi = \frac{1}{2}(\phi_1 - \phi_2) \tag{6.103}$$

Now let $\beta_r = r_r e^{i\phi_r}$, $\beta_s = r_s e^{i\phi_s}$, then

$$\beta_r^* A + \beta_s^* A^\dagger = \frac{r_r e^{-i\phi_r}}{\sqrt{|r_r|^2 - |r_s|^2}} A + \frac{r_s e^{-i\phi_s}}{\sqrt{|r_r|^2 - |r_s|^2}} A^\dagger. \tag{6.104}$$

$$\tag{6.105}$$

Comparing with (6.100), and using (6.103), we see that we can write

$$\beta_r^* A + \beta_s^* A^\dagger = S_A^\dagger(\xi) A S_A(\xi) \tag{6.106}$$

where $\xi = re^{i\phi}$ with

$$r = \cosh^{-1}\left(\frac{r_r}{\sqrt{|r_r|^2 - |r_s|^2}}\right) = \sinh^{-1}\left(\frac{r_s}{\sqrt{|r_r|^2 - |r_s|^2}}\right), \tag{6.107}$$

$$\phi = \frac{1}{2}(\phi_r - \phi_s). \tag{6.108}$$

Recall that if $|\phi\rangle$ is an equilibrium solution, then $(\beta_r^* A + \beta_s^* A^\dagger)|\phi\rangle = 0$ can be rewritten as

$$S_A^\dagger(\xi) A S_A(\xi)|\phi\rangle = 0 \qquad \Rightarrow \qquad S_A(\xi)|\phi\rangle = |0\rangle_A|\psi\rangle_B \tag{6.109}$$

where $|n\rangle_A|m\rangle_B$ denotes the representation of the system's state in terms of combined bosonic modes, and $|\psi\rangle_B$ is an arbitrary state in the Hilbert space defined by annihilation operator B. In the process of this derivation, we have used the fact that if a state $|\psi\rangle$ satisfies $A|\psi\rangle = 0$, then $|\psi\rangle$ must be in the ground state with respect to the mode that corresponds to annihilation operator A.

Thus, we find the steady state to be a squeezed state in the Hilbert space of A

$$|\phi\rangle = S_A(-\xi)|0\rangle_A|\psi\rangle_B \tag{6.110}$$

In order to define the solution completely, we will need to consider the second condition.

The Condition $(B^\dagger A - A^\dagger B)|\phi\rangle = 0$

: Substituting the partial solution into equation (6.98), we obtain

$$(B^\dagger A - A^\dagger B) S_A^\dagger(\xi)|0\rangle_A|\psi\rangle_B = 0. \tag{6.111}$$

Now, using the commutation properties of the squeezing operator,

$$\begin{aligned} A S_A^\dagger &= S_A^\dagger(\xi)\left[A\cosh(r) + A^\dagger e^{-2i\phi}\sinh(r)\right], & (6.112)\\ A^\dagger S_A^\dagger &= S_A^\dagger(\xi)\left[A e^{-2i\phi}\sinh(r) + A^\dagger\cosh(r)\right], & (6.113) \end{aligned}$$

we can rearrange the equation so that the A operators act on the steady state first. This allows us to use the fact that the steady state is the vacuum state of mode A, and thus $A|\phi\rangle = 0$, to simplify the equation considerably:

$$\begin{aligned} & S_A^\dagger\left\{\left[A\cosh(r) + A^\dagger e^{-2i\phi}\sinh(r)\right]B^\dagger + \left[A e^{-2i\phi}\sinh(r) + A^\dagger\cosh(r)\right]B\right\}|0\rangle_A|\psi\rangle_B \\ = \;& S_A^\dagger A^\dagger\left[\cosh(r)B + e^{-2i\phi}\sinh(r)B^\dagger\right]|0\rangle_A|\psi\rangle_B \\ = \;& S_A^\dagger(\xi) S_B^\dagger(\xi) B S_B(\xi) A^\dagger|0\rangle_A|\psi\rangle_B = 0. \end{aligned} \tag{6.114}$$

The above equation is only true when

$$\begin{aligned} B S_B(\xi)|0\rangle_A|\psi\rangle_B &= 0 & (6.115)\\ \Rightarrow \quad S_B(\xi)|\psi\rangle_B &= |0\rangle & (6.116)\\ \Rightarrow \quad |\psi\rangle_B &= S_B(-\xi)|0\rangle. & (6.117) \end{aligned}$$

The last step comes about from the property $S^{-1}(\xi) = S(-\xi)$. Thus, we obtain an explicit form for the steady state solution of the cascaded system as

$$|\phi\rangle = S_A(-\xi) S_B(-\xi)|0\rangle_A|0\rangle_B, \tag{6.118}$$

where S_A and S_B are the squeezing operators with respect to A and B respectively, and $\xi = re^{i\phi}$ is defined by (6.107) and (6.108). Therefore, we have obtained an explicit solution for the steady state in the limit where n tends to infinity. However, it is not written in a particularly elucidating form, since it involves operators A and B that have no direction physical interpretation.

In order to make physical interpretations, we will need to make a change of coordinates back to the separated atomic modes that are defined by annihilation operators a_1 and a_2. First of all, it is clear that if there are no atoms in the $|1\rangle$ state in any modes, then no mater what basis you choose, there still won't be any atoms in the $|1\rangle$ state. Thus, $|0\rangle_A|0\rangle_B = |0\rangle_1|0\rangle_2$.

It is less obvious, but also true, that the squeezing operators are also preserved under the change of basis. Explicitly

$$S_A(\xi)S_B(\xi) = e^{\frac{1}{2}\xi^*(A^2+B^2)-\frac{1}{2}\xi((A^\dagger)^2+(B^\dagger)^2)}, \tag{6.119}$$

but $A^2 + B^2 = a_1^2 + a_2^2$, and hence

$$S_A(\xi)S_B(\xi) = S_1(\xi)S_2(\xi). \tag{6.120}$$

And thus, our steady state is of the form

$$|\phi\rangle = S_1(-\xi)S_2(-\xi)|0\rangle_1|0\rangle_2. \tag{6.121}$$

The above equation verifies our conjecture. It is a product state, and hence is not entangled. Therefore, we can formally state

Proposition 13 (Entanglement Decay for Non-Resonant Many-Atom Systems) *Consider a cascaded cavity system that is non-resonant. In the limit that the number of atoms within each cavity tends to infinity, the entanglement of its steady state tends to zero.*

6.8 The Cascaded Cavity System as an Entanglement Source

In this chapter, we have managed to characterize completely the behaviour of entanglement in the Cascaded Cavity System. In particular, while the system is capable of generating perfect entanglement for the case of cascaded qubits, this ideal behaviour does not scale well with the number of atoms.

The cause of this, in a nutshell, is quite simple. Suppose Alice attempts to transmit information to Bob with a pack of cards. To make full use of the deck, she could associate with each of the 52 cards a unique piece of information, and hence the delivery of each card would represent $\lg 52$ bits of information. Now if Bob was innumerate, and also too stupid to tell the difference between the shapes that represent the four suits,[10] then Alice's only choice would to encode information only on whether a card is black of red, reducing the transfer rate to 1 bit per card.

[10] Bob, for instance, could be Alice's pet rabbit.

The inefficiency of our entanglement procedure stems from an analogous source. The nature of the entanglement is a correlation between odd and even states of the system, that is, the state of one subsystem is highly dependent on whether the other subsystem has an odd or even number of atoms in $|1\rangle$, but not significantly on exactly how many atoms are in $|1\rangle$. This implies that while that the entanglement could be perfect in the case of qubits, as there is only one odd and one even state, not much extra entanglement was introduced when we increased the number of states available to each cavity.

Therefore, while the steady state of the cascaded cavity system is an excellent protocol for creating entanglement between qubits, and useful for entangling other low dimensional atomic systems, it is unlikely to have the potential to entangle high dimensional systems. If we were interested in creating entanglement between qudits of very large dimensions, we will need to consider an alternative approach.

Chapter 7

Generation of Many Atom Entanglement

"There is nothing more practical than a good theory. "
-*Leonid Ilich Brezhnev*

7.1 Introduction

In the previous chapter, we saw that generating entanglement for high dimensional atomic systems cannot be achieved satisfactorily by simply increasing the number of atoms within each cavity. In this section, we explore how we can modify the cascaded cavity system so that it will generate ideal entanglement in the limit of a very large number of atoms.

7.2 The Bosonic Analogy

When we are to design a highway bridge so that it does not collapse in the limit of sustained peak hour traffic, we do so by applying our model to the limiting condition first, and then from the model, derive the parameters required to construct the bridge. The design of an entanglement generator is similar; if we want to generate satisfactory entanglement between large dimension atomic qudits, then we will need to consider first the limiting case.

Hence, our approach will be to investigate the case of the many atom limit first, and from that deduce a set of appropriate parameters. The system can then be analyzed for how it scales down to lower dimensions.

7.2.1 Entanglement and Two-Mode Squeezing

Before we can design a device that will produce significant entanglement in the limiting case, it is helpful to know exactly what state we are trying to generate. Consider two infinite dimensional qudits and suppose we wished them to exhibit optimal entanglement, then the obvious candidate state is the generalization of the maximally entangled state of two qudits

$$|\phi\rangle = \lim_{n\to\infty} |\phi_n\rangle = \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{n=0}^{N} |n\rangle|n\rangle. \tag{7.1}$$

If we consider the above to be in the Fock state basis, then this state represents a perfectly squeezed two mode squeezed state. That is, from (2.17),

$$|\phi(\lambda)\rangle = \lim_{\lambda\to 1} \sqrt{1-\lambda^2} \sum_{n=0}^{\infty} \lambda^n |n\rangle_A |n\rangle_B. \tag{7.2}$$

Now recall that in the many atom limit, provided certain assumptions are made about the states available to the system, we can rewrite the system in terms of bosonic operators. Thus, the problem of designing a variation of the cascaded cavity system that does produce optimal entanglement in the case of many atoms is reduced to a search for a cascaded bosonic system with the same algebraic representation.

7.2.2 The Many Atom Approximation Revisited

From previous chapters, we know already that the steady state will not be a good candidate for generating many atom entanglement. That is, in the case where the cavity systems are non-symmetric, the steady state is not pure, and in the case where they are symmetric, the entanglement is not optimal. This implies that if we wish to achieve better entanglement between two cavity systems with many atoms, we will need to look at transient behaviour.

In order to avoid loss of generality, we consider the evolution of a completely general cascaded cavity system, such that the parameters of the two cavities need not be equal. Recall that the dynamics of such a system is given by (5.20), reprinted here for convenience,

$$\begin{aligned}\dot{\rho} = &\sum_{i=1,2} \left(2R_i\rho R_i^\dagger - R_i^\dagger R_i\rho - \rho R_i^\dagger R_i\right) \\ &+ 2\sqrt{\epsilon}(\rho R_1^\dagger R_2 - R_2\rho R_1^\dagger + R_2^\dagger R_1\rho - R_1\rho R_2^\dagger),\end{aligned} \tag{7.3}$$

where the operator

$$R_k = \beta_{r,k}^* S_k + \beta_{s,k}^* S_k^\dagger, \qquad k = 1,2 \tag{7.4}$$

is associated with the cavity photon emission and the change of state of one atom.

Now, if we know that all atoms in a cavity are in $|1\rangle$ at time $t = 0$, there should be an interval of time $t < t_c$ such that the vast majority of atoms within the cavity remain in $|1\rangle$, provided n is large. Hence, let us assume that both cavities begin with all atoms in $|1\rangle$, i.e:

$|\phi(0)\rangle = |\frac{n}{2}\rangle|\frac{n}{2}\rangle$, then for all $t < t_c$, we can make the Schwinger approximations as was done in the previous chapter,

$$S_i = \sqrt{n}b_i^\dagger \tag{7.5}$$

$$S_i^\dagger = \sqrt{n}b_i, \qquad S_{z,i} = \frac{n}{2}. \tag{7.6}$$

Note that the assumption that all atoms start in $|1\rangle$ is completely arbitrary, since making the alternative assumption would simply result in an interchange of the parameters $\beta_{r,i}$ and $\beta_{s,i}$.

Now that we have written the cascaded system in terms of bosonic operators, we consider a bosonic system that shares the dynamics of the above equation.

7.2.3 The Motion-light Entanglement Distributor

Peng and Parkins[45] proposed in 2002 a system to entangle the motional states of two atoms harmonically confined inside separate optical cavities. Like the cascaded cavity system considered extensively in this thesis, the cavities are linked by a unidirectional coupling, and the cavity fields together with auxiliary lasers incident through the sides of the cavities drive Raman transitions between neighbouring vibrational levels of each individual atom.

As we only wish to consider this system as a reference, we will refer the reader to [45] for a derivation of the master equation describing the atomic motional states. This master equation is given by

$$\begin{aligned}\dot{\rho_a} = \Gamma_1 \left(2b_1^\dagger \rho_a b_1 - b_1 b_1^\dagger \rho_a - \rho_a b_1 b_1^\dagger\right) + \Gamma_2 \left(2b_2 \rho_a b_2^\dagger - b_2^\dagger b_2 \rho_a - \rho_a b_2^\dagger b_2\right) \\ +2\epsilon\sqrt{\Gamma_1\Gamma_2}\left[(\rho_a b_1 b_2 - b_2\rho_a b_1)e^{i\phi} + (b_2^\dagger b_1^\dagger \rho_a - b_1^\dagger \rho_a b_2^\dagger)e^{-i\phi}\right]\end{aligned} \tag{7.7}$$

where b_1 and b_2 are the annihilation operators corresponding to the quantized harmonic motion of each of the two atoms. Analysis of this master equation has shown that ρ evolves into a two-mode squeezed state of infinite squeezing in the limit of perfect coupling, when $\Gamma_r = \Gamma_s$.

Now let us compare the above equation with (7.3). We see that if we set $\beta_{s,1} = \beta_{r,2} = 0$, and define $r = \frac{\beta_{s,2}}{\beta_{r,1}}$, then the master equation for the cascaded cavity system (7.3) becomes isomorphic to the one given for the entanglement distributor (7.7). Explicitly, we have $R_1 = \beta_{r,1}^*\sqrt{n}b_1^\dagger$, $R_2 = \beta_{s,2}^*\sqrt{n}b_2$ and hence

$$\begin{aligned}\dot{\rho} &= n|\beta_{r,1}|^2\left[\left(2b_1^\dagger\rho b_1 - b_1 b_1^\dagger\rho - \rho b_1 b_1^\dagger\right) + |r|^2\left(2b_2\rho b_2^\dagger - b_2^\dagger b_2\rho - \rho b_2^\dagger b_2\right)\right] \\ &\quad +2n|\beta_{r,1}|^2\sqrt{\epsilon}\left(r^*\rho b_1 b_2 - r^* b_2\rho b_1 + r b_1^\dagger b_2^\dagger\rho - r b_1^\dagger\rho b_2^\dagger\right).\end{aligned} \tag{7.8}$$

Now, this equation is identical to the one above, with constants set to

$$|\beta_{r,1}|^2 = \Gamma_1, \qquad |\beta_{r,2}|^2 = \Gamma_2, \qquad r = \sqrt{\frac{\Gamma_2}{\Gamma_1}}e^{-i\phi} \tag{7.9}$$

and an extra overall factor of n. Since overall factors do not change the dynamics of any system, they merely speed up the process by a factor of n, we see that the entanglement distributor and the cascaded cavity system with $\beta_{s,1} = \beta_{r,2} = 0$ will share identical qualitative dynamics. It has already been shown that the entanglement distributor generates perfect entanglement when $\Gamma_1 = \Gamma_2$. Hence, provided we set $r = 1$ with $\beta_{r,1}$, $\beta_{s,2}$ real, we will expect the cascaded cavity system that generates perfect entanglement in the limit of an infinite number of atoms; that is, we have the ideal system for generating entanglement between large scale collective states that we are looking for.

7.2.4 The Cascaded Cavity System with Unidirectional Pumping

What does it mean physically to have $\beta_{s,1} = \beta_{r,2} = 0$? Let us consider the first cavity. Recall that $\beta_{s,1}$ is simply a scaled rate for the transition $|0\rangle \rightarrow |1\rangle$. Thus, setting this value to zero implies that if any atom within the cavity is switched from $|1\rangle$ to $|0\rangle$, the result is final; there is no going back. Thus, we call this the cascaded system with unidirectional pumping.

Within the second cavity, we have $\beta_{r,2} = 0$. Thus, there is no way the external laser will cause a transition from $|1\rangle$ to $|0\rangle$. If this cavity has no input, then the atomic system would not evolve, as one recalls that we have initially set all states within the cavity to $|1\rangle$. However, the second cavity is coupled with the first, and hence there could be $|1\rangle$ to $|0\rangle$ transitions during evolution. At steady state however, we would expect all atoms to be driven back to the $|1\rangle$ state. Hence, without any need for analytical calculation, we can state that the steady state of this system is

$$|\phi_{ss}\rangle = |-\frac{n}{2}\rangle_1 |\frac{n}{2}\rangle_2. \tag{7.10}$$

From an intuitive perspective, as atoms cannot be pumped from the $|1\rangle$ state in the second cavity without external influence, we know that, initially, any deviation from the $|\frac{n}{2}\rangle_2$ state is caused by emission from the first cavity. Therefore, it is plausible that such an initial condition could cause the maximum correlations, and be a candidate for possible generation of entanglement.

7.3 Two-Mode Squeezing in the Bosonic Limit

With a promising proposal at hand, we are now in a position to analyze the entanglement generated by the system in the limit of infinite atom number. In this case, the Von Neumann measure, while exact, becomes less useful. It is just not very practical, at least numerically, to determine the eigenvalues of an infinite dimensional matrix.

7.3.1 Squeezing as an Entanglement Indicator

Suppose we were given a two-mode squeezed state, then we can employ our knowledge about the relation between such states and Von Neumann Entanglement. From Section 2.5.2, we

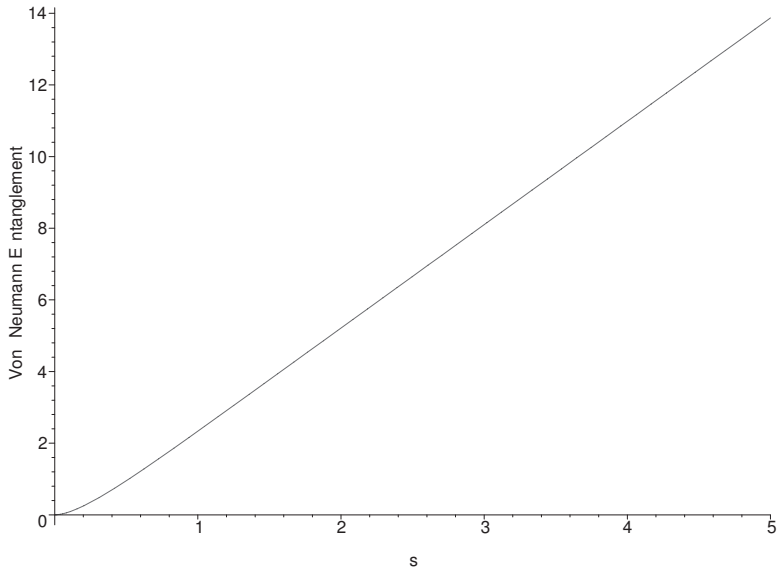

Figure 7.1: Plot of the Von Neumann entanglement in terms of the squeezing parameter s.

know that if we have a pure state with quadrature uncertainties

$$\left\langle (X_1 \pm X_2)^2 \right\rangle = 2e^{\pm 2s}, \qquad \left\langle (Y_1 \pm Y_2)^2 \right\rangle = 2e^{\mp 2s}, \tag{7.11}$$

where $X_i = b_i + b_i^\dagger$, $Y_i = -i(b_i - b_i^\dagger)$, then the resulting entanglement of the system is given by

$$E(|\phi\rangle) = \cosh^2(s)\lg(\cosh^2(s)) - \sinh^2(s)\lg(\sinh^2(s)). \tag{7.12}$$

Figure 7.1 demonstrates a plot of the Von Neumann entanglement against the squeezing parameter. One can see that the entanglement grows almost linearly with s, and is zero when $s = 0$, that is, when

$$\left\langle (X_1 \pm X_2)^2 \right\rangle = \left\langle (Y_1 \pm Y_2)^2 \right\rangle = 2, \tag{7.13}$$

signifying that the state is not squeezed. Therefore, whether a state is two-mode squeezed can be used as an indication and a quantitative measurement of the degree of entanglement of the system. In particular, a state is considered squeezed, and hence entangled, if

$$\left\langle (X_1 \pm X_2)^2 \right\rangle < 2 \qquad \text{or} \qquad \left\langle (Y_1 \pm Y_2)^2 \right\rangle < 2. \tag{7.14}$$

It can be shown that in the more general case where a given state is not necessarily a two-mode squeezed state, squeezing can still be used as an indicator of entanglement. In particular, Duan et. al. [52] demonstrated that if a state satisfies the inequality

$$\left\langle (X_1 \pm X_2)^2 \right\rangle + \left\langle (Y_1 \mp Y_2)^2 \right\rangle \leq 4 \tag{7.15}$$

then it cannot be factorized. Note, however, that the converse of the above is not necessarily true unless the state is Gaussian. Thus, while squeezing cannot be used as an exact measure of entanglement, it is often a good indication of entanglement in cases where the exact measure is difficult to apply.

7.3.2 Squeezing in the Unidirectionally Pumped Cascaded System

The ability to detect entanglement via two-mode squeezing is very convenient, given that we do not need to solve for the state of the system to determine the values of the observables $\langle (X_1 \pm X_2)^2 \rangle$ and $\langle (Y_1 \pm Y_2)^2 \rangle$. In fact the time evolution of the observables can be directly evaluated from the master equation

We first derive a closed set of differential equations for the correlation functions $\left\langle b_1^\dagger b_1 \right\rangle$, $\left\langle b_2^\dagger b_2 \right\rangle$ and $\langle b_1 b_2 \rangle$:

$$\frac{1}{n|\beta_{r,1}|^2}\left\langle \dot{b_1^\dagger} b_1 \right\rangle = 2\left(1 + \left\langle b_1^\dagger b_1 \right\rangle\right), \tag{7.16}$$

$$\frac{1}{n|\beta_{r,1}|^2}\left\langle \dot{b_2^\dagger} b_2 \right\rangle = -2|r|^2\left\langle b_2^\dagger b_2 \right\rangle + 2\sqrt{\epsilon}\left(r^* \langle b_1 b_2 \rangle + r\left\langle b_1^\dagger b_2^\dagger \right\rangle\right), \tag{7.17}$$

$$\frac{1}{n|\beta_{r,1}|^2}\left\langle \dot{b_1 b_2} \right\rangle = (1-|r|^2)\langle b_1 b_2 \rangle + 2r^*\sqrt{\epsilon}\left(\left\langle b_1^\dagger b_1 \right\rangle + 1\right). \tag{7.18}$$

As the atomic systems are initially in the $|\frac{n}{2}\rangle|\frac{n}{2}\rangle$ state, i.e, with no atoms in state $|0\rangle$, $\left\langle b_i^\dagger b_i \right\rangle = 0$ at $t = 0$. Applying this initial condition, we can solve the above set of equations to obtain

$$\left\langle b_1(t) b_1^\dagger(t) \right\rangle = e^{2n|\beta_{r,1}|^2 t}, \tag{7.19}$$

$$\left\langle b_1^\dagger(t) b_1(t) \right\rangle = e^{2n|\beta_{r,1}|^2 t} - 1, \tag{7.20}$$

$$\left\langle b_2^\dagger(t) b_2(t) \right\rangle = \frac{4\epsilon|r|^2}{(1+|r|^2)^2}\left(e^{n|\beta_{r,1}|^2 t} - e^{-n|\beta_{r,1} r|^2 t}\right)^2, \tag{7.21}$$

$$\langle b_1(t) b_2(t) \rangle = \frac{2\sqrt{\epsilon} r^*}{1+|r|^2} e^{n|\beta_{r,1}|^2 t}\left(e^{n|\beta_{r,1}|^2 t} - e^{-n|\beta_{r,1} r|^2 t}\right). \tag{7.22}$$

With all other correlations identically zero. It follows that

$$\begin{aligned} \left\langle X_1^2(t) \right\rangle = \left\langle Y_1^2(t) \right\rangle &= \left\langle b_1^\dagger(t) b_1(t) \right\rangle + \left\langle b_1(t) b_1^\dagger(t) \right\rangle \\ &= 2e^{2n|\beta_{r,1}|^2 t} - 1 \end{aligned} \tag{7.23}$$

$$\begin{aligned} \left\langle X_2^2(t) \right\rangle = \left\langle Y_2^2(t) \right\rangle &= \left\langle b_2^\dagger(t) b_2(t) \right\rangle + \left\langle b_2(t) b_2(t)^\dagger \right\rangle, \\ &= \frac{8\epsilon|r|^2}{(1+|r|^2)^2}\left(e^{n|\beta_{r,1}|^2 t} - e^{-n|\beta_{r,1} r|^2 t}\right)^2 + 1 \end{aligned} \tag{7.24}$$

$$\begin{aligned} \langle X_1(t) X_2(t) \rangle = -\langle Y_1(t) Y_2(t) \rangle &= \langle b_1(t) b_2(t) \rangle + \left\langle b_1^\dagger(t) b_2^\dagger(t) \right\rangle, \\ &= \frac{2\sqrt{\epsilon}}{1+|r|^2}(r + r^*) e^{n|\beta_{r,1}|^2 t}\left(e^{n|\beta_{r,1}|^2 t} - e^{-n|\beta_{r,1} r|^2 t}\right) \end{aligned} \tag{7.25}$$

Therefore, the variances of the sum and difference quadratures are given by

$$\left\langle (X_1(t) \pm X_2(t))^2 \right\rangle = \left\langle (Y_1(t) \mp Y_2(t))^2 \right\rangle = 2\left| e^{n|\beta_{r,1}|^2 t} \pm \frac{2\sqrt{\epsilon} r}{1+|r|^2}\left(e^{n|\beta_{r,1}|^2 t} - e^{-n|\beta_{r,1} r|^2 t}\right)\right|^2.$$

Consider the case of resonance, where $\beta_{r,1} = \beta_{s,2}$, and $\epsilon = 1$, then the above equation reduces to

$$\left\langle (X_1(t) - X_2(t))^2 \right\rangle = \left\langle (Y_1(t) + Y_2(t))^2 \right\rangle = 2e^{-2n|\beta_{r,1}|^2 t} \to 0 \qquad \text{as} \qquad t \to \infty. \tag{7.26}$$

Due to the equality between $\left\langle (X_1(t) - X_2(t))^2 \right\rangle$ and $\left\langle (Y_1(t) + Y_2(t))^2 \right\rangle$, the condition for entanglement can be simplified to

$$\left\langle (X_1 - X_2)^2 \right\rangle < 2 \tag{7.27}$$

At this point, it is useful to define a normalized parameter to indicate the degree of squeezing. That is, define a squeezing indicator $\mathcal{S}$, such that the system is squeezed, and hence entangled, if $\mathcal{S} < 1$. In this case, the obvious choice would be

$$\mathcal{S} = \frac{1}{2} \left\langle (X_1 - X_2)^2 \right\rangle < 1 \tag{7.28}$$

For the uni-directionally pumped cavity in the many atom limit,

$$\mathcal{S} = e^{-2n|\beta_{r,1}|^2 t} \tag{7.29}$$

That is, in the limit of ideal coupling, large atom number, and with equal and opposite pumping rates in each of the two cavities, we get a state of perfect squeezing and thus maximal (and infinite) entanglement in the limit as n tends to infinity.

This demonstrates that our proposed system can indeed generate unconditional entanglement in the limit of large atomic clouds trapped within cascaded cavities.

7.3.3 Parameter Dependence and Non-resonant Behaviour

When r is not exactly one, we would expect the squeezing to become less than optimal. A demonstration of how the squeezing evolves is shown in Figure 7.2.

One can see that unless r is exactly unity, both quadrature variances tend to infinity. However, provided we set $r > 0.5$, there are some values of time where the system is entangled. Note that this system is not symmetric with respect to r, since the cavities are not symmetric.

Since in practice, perfect resonance can never be achieved, this model suggests that we should stop the system at the point where it achieves optimal squeezing. For the case where r is real, this time can be evaluated analytically using simple calculus:

$$t_{opt} = \frac{1}{n(|\beta_{r,1}|^2 + |\beta_{s,2}|^2)} \ln\left(\frac{1 + |r|^2}{1 + |r|^2 - 2r\sqrt{\epsilon}} \right). \tag{7.30}$$

Note that in the limit where $r \to 0$ or ∞, $t_{crit} \to 0$, and thus no squeezing is achieved at any time t. Also in the case of perfect coupling ($\epsilon = 1$), as r approaches unity, the argument of the logarithm diverges to infinity, which conforms with earlier analysis.

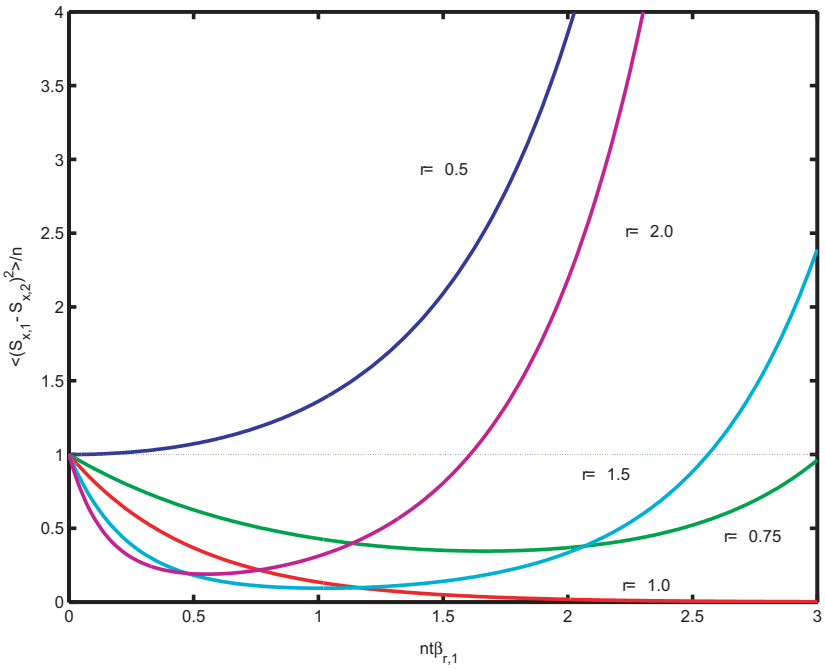

Figure 7.2: Plot of the squeezed quadrature variance $\left\langle (S_{x,1}(t) - S_{x,2}(t))^2 \right\rangle$ for various values of r. Note that optimal squeezing is achieved when $r = 1$ if we assume that the system can evolve indefinitely. Note that there are certain times for which systems with $r > 1$ exhibit better squeezing than with $r = 1$.

7.4 Spin Squeezing

Even though our system might contain a very large number of atoms, this number will still be finite. The simple fact that there is a finite limit to the energy levels of the collective atomic modes does introduce some significant differences. After all, the bosonic limit is still only an approximation.

Ideally, we would like to analyze the case of finite atom number by performing some sort of generalization from the bosonic limit. Not only will this save valuable research time and the reader many headaches, it also allows us to see the link between the cascaded atomic system and its bosonic limit explicitly.

While it is usual to characterize entanglement of continuous variable states by two mode squeezing, the concept can be generalized to the case of spin states. Recall that in the bosonic case, a state is considered squeezed, and hence entangled, if (but not only if) it satisfies the condition

$$\left\langle (X_1(t) - X_2(t))^2 \right\rangle < 2, \tag{7.31}$$

where

$$X_i = b_i + b_i^\dagger, \qquad i = 1, 2. \tag{7.32}$$

But in the bosonic limit of the collective atomic system, this operator is proportional to the equivalent angular momentum of the cavity with respect to the x direction. That is,

$$S_{x,i} = \frac{1}{2}\left(S^\dagger + S\right) \approx \frac{\sqrt{n}}{2} X_i. \tag{7.33}$$

Hence the squeezing indicator can be rewritten in the form

$$S = \frac{2}{n}\left\langle (S_{x,1}(t) - S_{x,2}(t))^2 \right\rangle. \tag{7.34}$$

While this reformulation still uses the bosonic approximation, it has the advantage of including explicit dependence on n within the model. We will use this form primarily for analytical calculations involving cavities with a large number of atoms.

7.4.1 Spin Squeezing in the General Case

While the above formula is ideal when the bosonic limit does hold, we would also like to run numerical simulations where this limit is not necessarily true. In this case, we cannot assume that $\left\langle b^\dagger b \right\rangle \ll n$ for either atomic system.

We can derive the conditions for spin squeezing by considering the commutation relation for spin that is analogous to the bosonic relation $[X, Y] = 1$. That is

$$[S_x, S_y] = iS_z, \tag{7.35}$$

and hence we have

$$[S_{x,1} - S_{x,2}, S_{y,1} - S_{y,2}] = i\left(S_{z,1} + S_{z,2}\right) \tag{7.36}$$

with Heisenberg uncertainty relation[1]

$$\left\langle (S_{x,1} - S_{x,2})^2 \right\rangle \left\langle (S_{y,1} - S_{y,2})^2 \right\rangle \leq \frac{1}{4}\left|\langle S_{z,1}\rangle + \langle S_{z,2}\rangle\right|^2. \tag{7.37}$$

Hence, we can consider a state squeezed if

$$\left\langle (S_{x,1} - S_{x,2})^2 \right\rangle \leq \frac{1}{2}\left|\langle S_{z,1}\rangle + \langle S_{z,2}\rangle\right|. \tag{7.38}$$

Therefore we can define the squeeze indicator for the generalized spin system by

$$S = \frac{2\left\langle (S_{x,1} - S_{x,2})^2 \right\rangle}{\left|\left\langle S_{z,1}\right\rangle + \left\langle S_{z,2}\right\rangle\right|}. \tag{7.39}$$

We note that this is a simple generalization of (7.34), whereby we can no longer regard the commutator $[S_x, S_y]$ to be a constant. The biggest qualitative difference this presents is that all squeezing will be destroyed if half of the atoms have flipped states (i.e $\langle S_{z,1}\rangle = \langle S_{z,2}\rangle = 0$), as the expectation value of the commutator becomes zero.

[1] We can write the relation is form since the first order terms $\langle S_{x,i}\rangle = \langle S_{y,i}\rangle = 0$.

7.4.2 Spin Squeezing as an Indicator for Entanglement

We have already demonstrated that two mode squeezing can be used as an indicator for entanglement in the case of continuous variables. Can a generalization be made for states of finite dimensions? The answer is yes, provided one employs enough mathematical rigour. Explicitly, it has been demonstrated that any state that satisfies the relation [46]

$$\left\langle (S_{x,1} - S_{x,2})^2 \right\rangle + \left\langle (S_{y,1} + S_{y,2})^2 \right\rangle < | \langle S_{z,1} \rangle + \langle S_{z,2} \rangle | \tag{7.40}$$

is entangled. Assuming, we have $\left\langle (S_{x,1} - S_{x,2})^2 \right\rangle = \left\langle (S_{y,1} + S_{y,2})^2 \right\rangle$ as demonstrated by numerical results, the condition is reduced to exactly $S < 1$, the condition for squeezing. Note that we have once again included the symmetry condition $\langle S_{x,i} \rangle = \langle S_{y,i} \rangle = 0$, which is demonstrated explicitly by Berry and Sanders [47].

Thus, we have cast the problem of finite dimensional entanglement into a form algebraically similar to that of the bosonic case. That is, we can use the squeezing indictor S as a measure of entanglement.

7.5 Spin Squeezing Generation

With a simple way to detect entanglement at hand, we are ready to extend our analysis of the cascaded cavity system with unidirectional pumping to the more realistic case where the number of atoms within each cavity is finite. Due to the non-constant commutator $[S_x, S_y]$, it is fairly evident that an analytical solution for the general case would be too optimistic.

Rather, we shall try another approach, i.e, we reconsider the bosonic limit. As with any approximation, we need to determine the region in which it is valid, and what occurs in regions where it is invalid. By determining the region of validity, we can get a good estimate on how entanglement depends on the number of atoms within each cavity.

7.5.1 Quadrature Squeezing Dependence on Atom Number

Recall that in taking the bosonic limit, we had made the assumption that it is only valid for some time interval $t < t_c$ in which the majority of the atoms are still in the $|1\rangle$ state. Here, t_c indicates some estimate of the critical time where this assumption fails.

Now $\left\langle b_1^\dagger b_1 \right\rangle$ and $\left\langle b_2^\dagger b_2 \right\rangle$ represent the number of atoms within each cavity that are not in the $|1\rangle$ state. Thus, this assumption is equivalent to

$$\left\langle b_1^\dagger b_1 \right\rangle, \left\langle b_2^\dagger b_2 \right\rangle \ll n. \tag{7.41}$$

Fortunately, these expectation values are easy to calculate, and their time evolution was already evaluated in the process of determining how squeezing evolved within the system. Equations (7.19) and (7.21) predict that the number of atoms in $|1\rangle$ for both cavities tends

towards infinity as t tends to infinity, which clearly indicates that t_c exists, and is finite. That is, our bosonic limit will only be accurate for a finite time.

In particular, our previous analysis has shown that infinite squeezing is only achieved after an infinite amount of time (7.30), and hence we will not be able to generate such squeezing with a finite number of atoms. Thus, it is necessary to consider how squeezing is affected by finite atom numbers, and how it improves as we approach the infinite limit.

We can define a specific value for t_c by introducing a tolerance $\frac{1}{c}$, defined to be the maximum proportion of atoms in$|0\rangle$ that out model is valid for. In which case, we have inequalities

$$\left\langle b_1^\dagger(t)b_1(t)\right\rangle, \left\langle b_2^\dagger(t)b_2(t)\right\rangle < \frac{n}{c}, \tag{7.42}$$

and so

$$t < \frac{1}{2n|\beta_{r,1}|^2}\ln\left(\frac{n}{c}\right) = t_c. \tag{7.43}$$

Consider a resonant system with perfect coupling, where $\left\langle (S_{x,1}(t) - S_{x,2}(t))^2\right\rangle$ is a monotonically decreasing function of time. We can now approximate the maximal amount of squeezing that can be achieved by considering the squeezing generated at the time t_c, when the model is no longer valid. At $t = t_c$, the variance of the squeezed quadrature is given by

$$\begin{aligned}
\left\langle (S_{x,1}(t_c) - S_{x,2}(t_c))^2\right\rangle &= \frac{n}{2}\left|e^{n|\beta_{r,1}|^2 t}\left(e^{n|\beta_{r,1}|^2 t_c} - e^{-n|\beta_{r,2}|^2 t_c}\right)\right|^2 && (7.44)\\
&= n\left|\left(\frac{n}{c}\right)^{\frac{1}{2}} - \left[\left(\frac{n}{c}\right)^{\frac{1}{2}} - \left(\frac{n}{c}\right)^{-\frac{1}{2}}\right]\right|^2 && (7.45)\\
&= c. && (7.46)
\end{aligned}$$

Notice that the minimum variance, to a first-order approximation, is dependent solely on c, our choice of tolerance. That is, suppose that we expected our bosonic limit to hold only when a small fraction of atoms have switched to $|0\rangle$, then c would be large, and consequently there would be little squeezing, and vice versa.

We can also evaluate the squeezing indicator

$$\mathcal{S}(t_c) = \left\langle (X_1(t_c) - X_2(t_c))^2\right\rangle = \frac{2c}{n} \sim \frac{1}{n} \tag{7.47}$$

Without having to know exactly where the bosonic limit breaks down, our analysis has already produced a useful result. The squeezing achieved for a finite number of atoms improves linearly with the number of atoms. In the limit where $n \to \infty$, we have infinite squeezing, and infinite entanglement, in accordance with the bosonic limit.

7.5.2 Regions of Validity for the Bosonic Limit

We have seen that by assuming the model fails after a certain fraction of atoms have switched atomic states, we are able to derive the qualitative dependence of squeezing on atom number. For a quantitative result, we will need to be able to estimate at what exact point does the model fail, i.e, obtain an estimate for c. Such a limit can be obtained with numerical methods

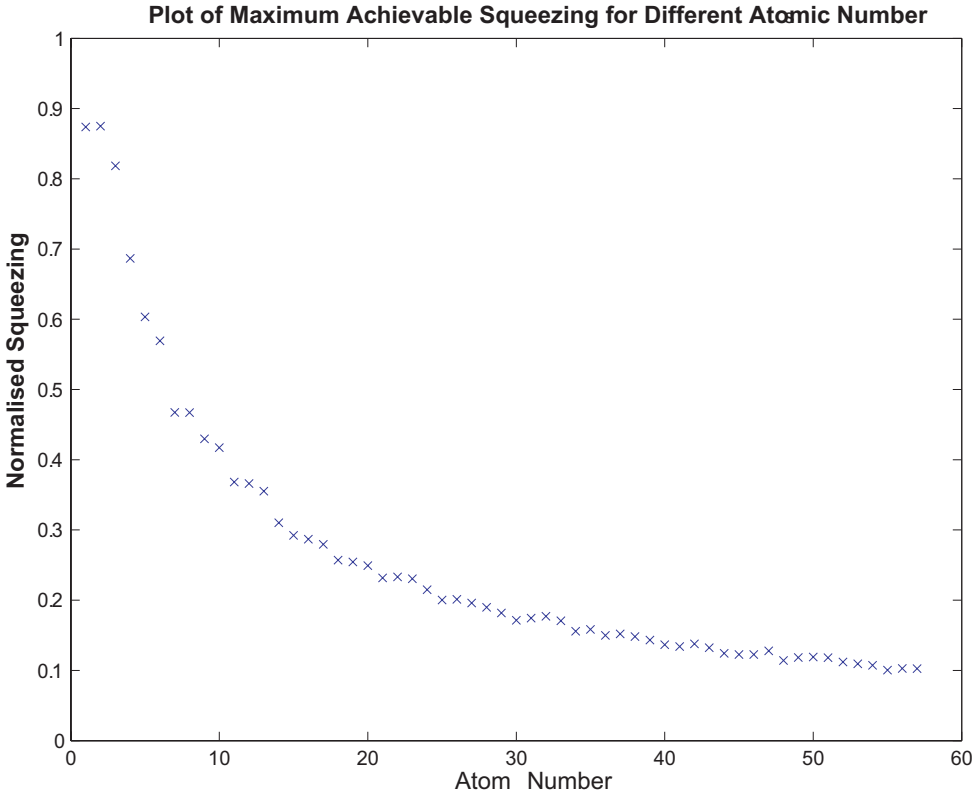

Figure 7.3: A plot of the best possible squeezing S for various atomic numbers up to $n = 57$. The results were produced using quantum trajectory simulations with 600 trajectories for each point. As predicted by theory, the degree of squeezing improves as the number of atoms is increased.

In the finite atom number case, the squeezing indicator is given by

$$\mathcal{S}(t) = 2\frac{\left\langle (S_{x,1}(t) - S_{x,2}(t))^2 \right\rangle}{|\langle S_{z,1}(t) + S_{z,2}(t) \rangle|}, \tag{7.48}$$

which reduces to (7.47) in the infinite-n approximation.

We use quantum trajectories to simulate the evolution of the cascaded cavity system for $1 < n < 60$, which allows us to find the minimum value of $\mathcal{S}$ for each n. The results are plotted in Fig. 7.3. The advantage of using the trajectory approach is that we can bypass the ambiguities of entanglement given for mixed states since each individual trajectory remains pure.

If $\mathcal{S} \sim \frac{1}{n}$ for large n, then we would expect $n\mathcal{S}$ to approach a constant limit, namely $2c$. That is, an estimate of c can be made by considering

$$\lim_{n\to\infty} n\mathcal{S} = 2c. \tag{7.49}$$

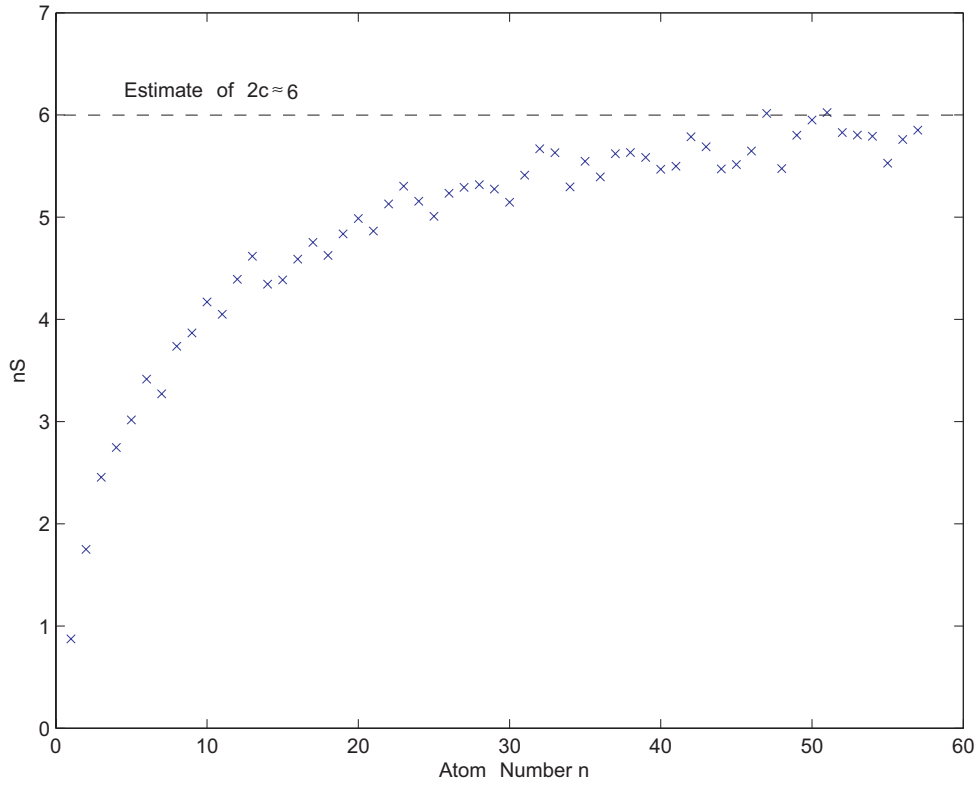

Figure 7.4: The limit of $n\mathcal{S}$ gives us a rough estimate for the parameter c. Thus, a plot of the first 57 atoms gives the estimate of $2c \approx 6$, or $c \approx 3$.

From (7.4) a preliminary estimate gives $c \approx 3$. While this result is rough, we will see in the following section that it can be used to optimise the amount of squeezing for low atom numbers.

7.5.3 Squeezing Enhancement for Finite Atom Numbers

Careful observation of Fig. 7.2 shows that if we restrict ourselves to finite times, $r = 1$ does not give the best possible squeezing. That is, at some time t_c, there exists some value of $r_{opt} > r$ such that the quadrature variance is less than it would be had the two transition rates been equal.

Figure 7.5 demonstrates this for the case where there are $n = 10$ atoms within each cavity. By increasing the value of r from 1.0 to 1.5, squeezing is improved by roughly 60%. Therefore, seeking optimal ratios between the Raman transition rates of the two cavity systems will be very important for achieving high levels of squeezing.

We can estimate the optimal value of r by assuming that optimal squeezing is achieved when the time at which the quadrature difference attains its minimum coincides with the time

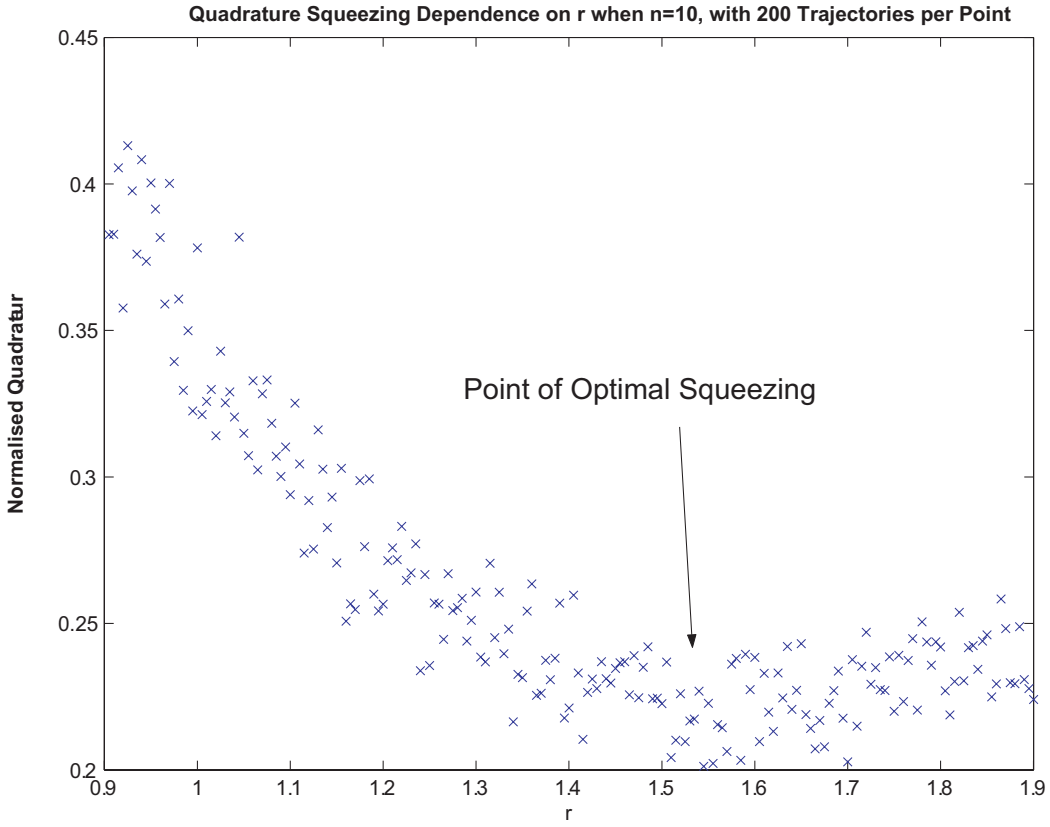

Figure 7.5: A plot of the minimum of the squeezed quadrature $\left\langle (X_{x,1}(t) - X_{x,2}(t))^2 \right\rangle$ for various r with atom number $n = 10$. In contrast to the case of $n = \infty$, $r = 1$ does not give the optimal squeezed state. Instead, maximum squeezing is achieved significantly far from resonance, when $r \approx 1.5$.

where the infinite-n approximation breaks down. Explicitly, for fixed n,

$$t_c = t_{opt} \tag{7.50}$$

$$\frac{1}{2}\ln\left(\frac{n}{c}\right) = \frac{1}{|r|^2+1}\ln\left(\frac{1+|r|^2}{1+|r|^2-2r}\right) \tag{7.51}$$

can be solved to estimate the value of r that will generate maximum entanglement. This equation is transcendental, but can be solved via numerical or graphical methods.

Fig. 7.6 illustrates the intersection of nt_c with nt_{opt} for several values of n. As $n \to \infty$, the optimal value of r, $r_{opt} \to 1$ as expected. However, for finite n, maximum entanglement occurs when $\beta_{s,2} > \beta_{r,1}$.

For example, consider the case where $n = 10$. Fig. 7.6 indicates that maximum entanglement occurs when $r \approx 1.6$. In Fig. 7.5, numerical simulation has shown that optimal entanglement occurred at $r \approx 1.5$, which is in good agreement.

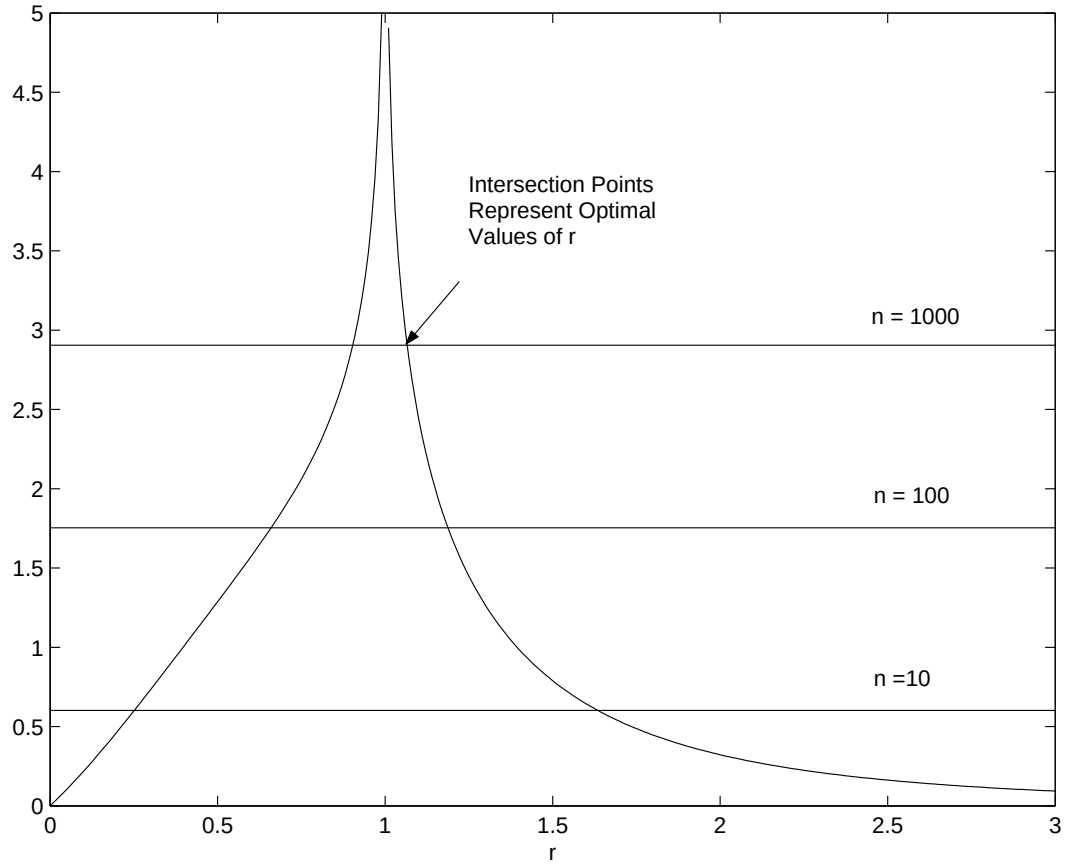

Figure 7.6: We can estimate the optimal value for r by considering the intersection of the time for optimal squeezing and the time where our large atom number approximation fails.

7.6 Implications

In this chapter, we have proposed a physical system that is capable of generating entanglement between qudits of higher dimensional systems. The particular advantage of the proposal is that the entanglement efficiency of the system increases with its dimension, and in the infinite dimensional limit, is only bounded by coupling inefficiencies and other loss mechanisms.

This method of entanglement generation also offers an elucidating contrast to the results given for entanglement at steady state for a symmetric cavity. In particular, it is interesting to observe that the latter was not effective at creating entanglement in high dimensional systems since it did not utilize those higher dimensions. In contrast, entanglement generated in the unidirectionally pumped system degrades in lower dimensional systems, since it relies on systems with a large number of states to approximate two-mode squeezing.

Together the two methods complement each other well, providing techniques to effectively generate entanglement in various different dimensions. This flexibility provided by the cas-

caded cavity system could potentially make it a promising implementation for quantum information processing and other related protocols.

Chapter 8

Conclusions

> "I may not have gone where I intended to go, but I think I have ended up where I intended to be. "
> -*Douglas Adams*

During the process of writing this thesis, there were a number of different objectives. From the practical perspective, the majority of protocols in the field of quantum information processing utilize an EPR-type source, a source of entanglement. There is no restriction on the form of this entanglement, it may be between the polarization states of photons or the energy states of atoms, and in this sense, entanglement can be regarded as a physical resource analogous to energy.

In particular, we know that coupling multiple systems storing in quantum information is an exceptionally difficult procedure, and thus generation of entanglement between qudits of higher dimensions may be a promising way to alleviate the problem. Thus, the applied side of this research was to investigate how variations of the cascaded cavity system could be utilized to entangle multidimensional quantum states.

From the more fundamental perspective, entanglement is a fascinating physical phenomenon in its own right, a true departure from classical logic. Whenever entanglement exists between two systems, each system loses its independent local reality. To measure exactly how much independent local reality is lost is a non-trivial task, especially in the presence of classical uncertainty. Our more theoretical objective was to consider how entanglement can be characterized and measured meaningfully in the face of such ambiguity.

8.1 Entanglement Generation in Many Atom Systems

We considered the possibility of using the cascaded cavity system to entangle the collective spin states of two atomic ensembles. Two different approaches were investigated, each a generalization of a limiting case that is known to be capable of generating optimal entanglement.

The first approach was the cascaded qubit system, which was known to be capable of generating a Bell state in the steady state in the limit of resonance [27]. So it was natural to consider if such a system could create optimally entangled states for collective spin systems if additional atoms were added inside each cavity. By constructing a numerical model, we were able to gain insight into the symmetry properties of the system, which allowed a rigorous analytical solution to be developed. This, in particular, gave us a clear understanding of how entanglement was manifested in the steady state.

We showed that while entanglement persisted in higher dimensional systems, it was limited by the fact that it was manifested in the form of anti-correlation between states of different parities. In particular, the higher dimensions were not effectively used, and hence entanglement within the system did not scale well with increasing dimensionality.

In order to generate entanglement more efficiently between collective spin states of higher dimensionality, we considered a modification of the system. That is, a system that would behave as a perfect two mode squeezer in the limit where its dimensionality tended to infinity. We found that if we set transition rates $\beta_{s,1} = \beta_{r,2} = 0$, i.e, utilize unidirectional pumping, then the behaviour of the cascaded cavity system, for large atom number, is identical to that of a motion-light entanglement distributor, a system already known to generate perfect entanglement in the limit where the transition rates of both cavities are identical.

By using approximations, aided with the use of quantum trajectories for numerical simulations, we generalized the result to the cascaded cavity system. Although the cascaded system with unidirectional pumping could not generate a state of maximal entanglement, we demonstrated using numerical simulations and analytical approximations that squeezing improved linearly with atom number. In particular, for large spin systems, the entanglement generated is very close to ideal. Therefore, this particular system is a promising proposal for entangling collective spin states of very large dimensionality.

The two proposals together provide methods to generate entanglement for a wide spectrum of atom numbers. This entanglement can then be used as a resource for various protocols in quantum information. For example, it has been shown that finite atom entanglement in large dimensions has the potential to conditionally teleport continuous states with almost perfect fidelity [48].

8.2 The Local Reality of Entanglement in Open Systems

Consider a bipartite open system evolving freely in space, its output are collected perfectly in a box. At some time during its evolution process, the interactions within the system are switched off, so that the system is now static. Suppose now, the system and its box are separated and placed into separate spaceships. Alice and Bob, pilot one ship each, and travel away from each other at close to the speed light.

We are left with an interesting question, what is the entanglement of the open system? This, at first glance, appears to be a perfectly valid question. Any system, if we are given complete information, is described by a pure state, and every pure state has a well defined value of

entanglement. However, the problem is actually far less trivial than it first appears.

In general, every interaction between two objects entangles the objects, and as the system and the box have interacted in the past, the state of the system and the state the box are now intertwined. There is indeed a pure state in this problem, but it is now a pure state of the system and the box containing its output combined. It is no longer possible, nor correct, to talk about the 'state' of the box, because the box is no longer an independent entity.

From a certain perspective, the problem of measuring bi-partite entanglement in an open system is similar to that of analyzing tri-partite entanglement. To make matters more complicated, we do not have an explicit formulation of the pure state, since we do not know the state of the box. The second objective of the thesis is devoted to this problem.

Classical methods involve throwing away the box, and looking at the system alone. That is, disregarding Bob and studying the system purely from the perspective of Alice. Of course, Bob's box does contain information, and by discarding it, entropy is introduced within Alice's system. This entropy, in turn, is the source of ambiguity in our measure of entanglement.

Typical methods to determine the entanglement of such systems involved attempts at dealing with this ambiguity directly, by methods of approximations or lower bounds[16]–[19], [25]. However, the inherent loss of information prevents such measures from exhibiting the conservation properties normally associated with a well behaved physical resource. This is no surprise, for we are only looking at half of the picture. It is akin to measuring only the kinetic energy before and after an inelastic collision, resulting in the conclusion that energy is not being conserved.

In this thesis, we have explicitly demonstrated an alternative approach. While we cannot know the nature of the Hilbert space that characterizes the box, it is feasible to extract classical information about the output by the use of projective measurements. Quantum trajectories facilitate this approach, and allow us to not discard the box completely. In turn, this allows the system to retain its purity, and thus gives the ability to characterize its entanglement in an exact manner.

To demonstrate this approach, we applied it to the cascaded qubit system. Previously, it was assumed that the system evolved gradually into a pure state, whereby entanglement was generated. The exact behaviour at resonance was not well known, since the master equation at this point had degenerate eigenvalues and no well-defined steady state. Using quantum trajectories, we observed that at resonance, the system was actually oscillating between two completely entangled Bell states after the first photon emission. As both states in this quantum cycle were perfectly entangled, the use of quantum trajectories allowed us to extract perfect entanglement from the system at any time, as opposed to having to wait for steady state.

While this is a useful result in itself, it can also be regarded in a more remarkable light. Suppose the system Alice has in her ship was the cascaded qubit system with parameters set for resonance, that is the system oscillates between the states $\frac{1}{\sqrt{2}}(|11\rangle - |00\rangle\rangle$ and $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle\rangle$ in a stable quantum cycle. The density operator of such an oscillation is an equal superposition of the two Bell states, which has zero entanglement under any mixed state

entanglement measure.[1] Thus Alice would be unable to extract any entanglement from the system with local measurements and classical communication, regardless of when it was frozen. Without taking Bob into account, the cascaded qubit system is in a mixed state that can be considered separable.

Now consider Bob, who opens his box and counts the number of photons trapped within. By seeing whether the final photon count is even or odd, Bob is able to deduce exactly which pure Bell state Alice's system is in. Thus, the process of measurement unravels the density operator, and produces entanglement, despite the fact that the measurement is *not* causally connected to Alice's system(see Figure 8.1).

While it is tempting at this point to resolve the paradox by assuming the entanglement was always there in the first place, one notes that there are many other ways Bob could have chosen to measure the state in his box, each of which will produce different unravellings of the cascaded qubit system, yielding different values for entanglement. Entanglement has somehow managed to escape the confines of local reality. In fact, the entire evolutionary history of the open system itself is dependent on what measurement Bob chooses to make.

This thought experiment then demonstrates why the conventional approach; whereby the evolution of a density operator is calculated, and from that, one attempts to estimate the entanglement of the system; is doomed to ambiguity. In applying this approach, we have already assumed that the system's evolution at each point is well defined, that there is a parameter of entanglement that we can attribute to the system. However, in truth, the entanglement between an open system and its environment is very much like the two pieces of a Bell state, where each individual piece does not exhibit local reality, and any entanglement contained within them is undefined until the point of measurement.

In studying the nature of entanglement in the cascaded qubit system, we were able to explicitly demonstrate these non-classical traits of entanglement. At the same time, we have shown that one way to sidestep the ambiguity of such systems is by performing measurements that determine how such a system interacts with the environment. While it can be argued that this measure of entanglement shifts the ambiguity to our choice of external measurement, it still leads to a result that has a clear physical interpretation. In particular, the entanglement measured can be generated in experimental situations.

Perhaps then, a formalism that includes external measurements could be considered as the only physical approach. For this thought experiment demonstrates that entanglement in open systems do not exhibit local reality, and how can one measure and quantize the value of something that is not defined?

8.3 Future Possibilities

There are a number of possible extensions to the research detailed in this thesis. First of all, the proposals for many-atom entanglement assumed that the loss due to imperfect coupling

[1] This was demonstrated in Chapter 5.

is negligible, with the understanding that this loss would cause some deterioration of the results. This sacrifice was made so that we could obtain detailed analytical results, and reveal how the system worked, rather than being limited to numerical simulations. However, if we wished to model the results of actual experiments, coupling loss cannot be neglected, and hence one simple extension would be the use of extensive numerical methods to consider how the entanglement is affected by coupling loss. As with any theory, it would also be very good to see an experimental realization of the proposals that we have outlined.

In the case of the unidirectionally coupled system, much of the analysis was limited to numerical simulations using quantum trajectories and somewhat ad hoc approximations. A rigorous, analytical approach similar to what was done for the cascaded qubit system may yield further insight into the system. The squeezing that was measured was the expected squeezing, i.e, the average of all trajectories, but it remains possible that certain trajectories could exhibit even greater squeezing. The difficulty with this analysis is that unlike qubit systems, multi-dimensional systems do not reset after a single photon collapse, and hence it is difficult to develop a fully analytical solution. However, perhaps by studying the equations extensively, one can obtain a number of useful, qualitative insights.

A more fundamental goal would be to extend the use of quantum trajectories as a measure of entanglement. Here we studied how entanglement may be quantified by considering one particular measurement made on the environment. In practice, however, certain couplings with the environment cannot be measured realistically, such as the coupling loss between the two cavities. We simply do not have the apparatus to, for instance, detect the number of photons that was scattered by a Faraday isolator.

To measure entanglement within such systems in an experimentally realizable fashion, we cannot assume that we can measure the interaction with the environment caused by such a loss. Despite the ambiguity this introduces into our system, Wiseman [49] has shown that only certain unravellings of the density operators can be physically realized by any possible continuous measurement made on the environment. Therefore, one conceivable measure of entanglement for such systems, is taking the minimum of all physically realizable unravellings, which would be a more accurate lower bound for the entanglement intrinsic within such a system. An alternative formulation would be taking the maximum if all physically realizable unravellings. This instead, would measure the maximal entanglement that can be distilled from such a system when local measurements made to the external environment are taken into account. It would be interesting to see how such a measure behaves.

Finally, we have mentioned that when an open system evolves, it becomes inexorably entangled with the environment. This extra entanglement is the cause of the ambiguity when one attempts to measure the entanglement generated within the system itself. Essentially, we are analyzing multi-partite entanglement, a problem known to be hard. It is then speculative that perhaps the use of quantum trajectories can shed some light on this problem.

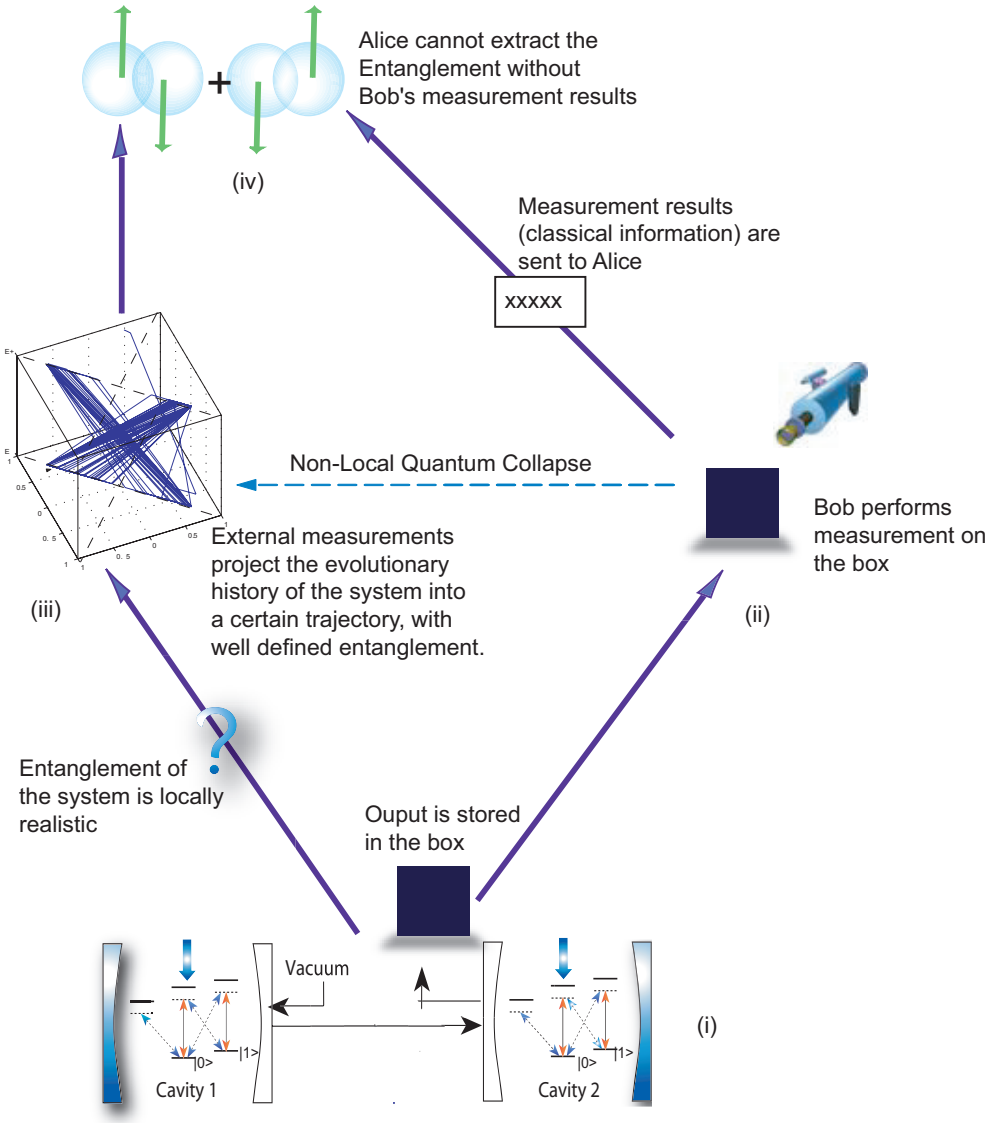

Figure 8.1: The output of the cascaded qubit system is stored in a box (i). Since the the system is entangled with the box, it does possess a local reality or well defined value of entanglement. Only when Bob performs a measurement on the box (ii) is the system projected into into a well defined quantum trajectory which gives rise to a well defined value of entanglement (iii). Causality is preserved since Alice cannot extract the entanglement in her system without knowledge of Bob's measurement.

Appendix A

Correlation Functions and the Quantum Regression Formula

> "I don't want to go into this in detail, but I would like to illustrate some of the tedium."
> *Anonymous Lecturer, Cambridge*

A.1 Introduction

In order to derive the master equation for the cascaded cavity system, we were required to simplify the expression

$$\begin{aligned}
\dot{\rho_m} = -\Bigg(& \int_0^\infty d\tau \mathrm{Tr}\left[a_1 e^{\mathcal{L}_c\tau}(a_1^\dagger \rho_c^{ss})\right](\hat{R}_1^\dagger \hat{R}_1 \rho_m - \hat{R}_1 \rho_m \hat{R}_1^\dagger) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_1 e^{\mathcal{L}_c\tau}(\rho_c^{ss} a_1^\dagger)\right](\rho_m \hat{R}_1 \hat{R}_1^\dagger - \hat{R}_1^\dagger \rho_m \hat{R}_1) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_1^\dagger e^{\mathcal{L}_c\tau}(a_1 \rho_c^{ss})\right](\hat{R}_1 \hat{R}_1^\dagger \rho_m - \hat{R}_1^\dagger \rho_m \hat{R}_1) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_1^\dagger e^{\mathcal{L}_c\tau}(\rho_c^{ss} a_1)\right](\rho_m \hat{R}_1^\dagger \hat{R}_1 - \hat{R}_1 \rho_m \hat{R}_1^\dagger) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_2 e^{\mathcal{L}_c\tau}(a_2^\dagger \rho_c^{ss})\right](\hat{R}_2^\dagger \hat{R}_2 \rho_m - \hat{R}_2 \rho_m \hat{R}_2^\dagger) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_2 e^{\mathcal{L}_c\tau}(\rho_c^{ss} a_2^\dagger)\right](\rho_m \hat{R}_2 \hat{R}_2^\dagger - \hat{R}_2^\dagger \rho_m \hat{R}_2) \\
& + \int_0^\infty d\tau \mathrm{Tr}\left[a_2^\dagger e^{\mathcal{L}_c\tau}(a_2 \rho_c^{ss})\right](\hat{R}_2 \hat{R}_2^\dagger \rho_m - \hat{R}_2^\dagger \rho_m \hat{R}_2)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty d\tau \mathrm{Tr}\left[a_2^\dagger e^{\mathcal{L}_c\tau}(\rho_c^{ss}a_2)\right](\rho_m\hat{R}_2^\dagger\hat{R}_2-\hat{R}_2\rho_m\hat{R}_2^\dagger)\\
&+\int_0^\infty d\tau \mathrm{Tr}\left[a_1 e^{\mathcal{L}_c\tau}(a_2^\dagger\rho_c^{ss})\right](\hat{R}_1^\dagger\hat{R}_2\rho_m-\hat{R}_2\rho_m\hat{R}_1^\dagger)\\
&+\int_0^\infty d\tau \mathrm{Tr}\left[a_1 e^{\mathcal{L}_c\tau}(\rho_c^{ss}a_2^\dagger)\right](\rho_m\hat{R}_2\hat{R}_1^\dagger-\hat{R}_1^\dagger\rho_m\hat{R}_2)\\
&+\int_0^\infty d\tau \mathrm{Tr}\left[a_2^\dagger e^{\mathcal{L}_c\tau}(a_1\rho_c^{ss})\right](\hat{R}_2\hat{R}_1^\dagger\rho_m-\hat{R}_1^\dagger\rho_m\hat{R}_2)\\
&+\int_0^\infty d\tau \mathrm{Tr}\left[a_2^\dagger e^{\mathcal{L}_c\tau}(\rho_c^{ss}a_1)\right](\rho_m\hat{R}_1^\dagger\hat{R}_2-\hat{R}_2\rho_m\hat{R}_1^\dagger)\\
&+\int_0^\infty d\tau \mathrm{Tr}\left[a_1^\dagger e^{\mathcal{L}_c\tau}(a_2\rho_c^{ss})\right](\hat{R}_1\hat{R}_2^\dagger\rho_m-\hat{R}_2^\dagger\rho_m\hat{R}_1)\\
&+\int_0^\infty d\tau \mathrm{Tr}\left[a_1^\dagger e^{\mathcal{L}_c\tau}(\rho_c^{ss}a_2)\right](\rho_m\hat{R}_2^\dagger)\hat{R}_1-\hat{R}_1\rho_m\hat{R}_2^\dagger)\\
&+\int_0^\infty d\tau \mathrm{Tr}\left[a_2 e^{\mathcal{L}_c\tau}(a_1^\dagger\rho_c^{ss})\right](\hat{R}_2^\dagger\hat{R}_1\rho_m-\hat{R}_1\rho_m\hat{R}_2^\dagger)\\
&+\int_0^\infty d\tau \mathrm{Tr}\left[a_2 e^{\mathcal{L}_c\tau}(\rho_c^{ss}a_1^\dagger)\right](\rho_m\hat{R}_1\hat{R}_2^\dagger-\hat{R}_2^\dagger\rho_m\hat{R}_1)+\dots\Bigg).
\end{aligned}
\tag{A.1}
$$

Noting the relation (4.54),

$$\langle X(t)Y(t+\tau)\rangle_{ss} = \lim_{t\to\infty}\langle X(t)Y(t+\tau)\rangle = \mathrm{Tr}(Ye^{\mathcal{L}_c\tau}(\rho X)), \tag{A.2}$$

$$\langle X(t+\tau)Y(t)\rangle_{ss} = \lim_{t\to\infty}\langle X(t+\tau)Y(t)\rangle = \mathrm{Tr}(Xe^{\mathcal{L}_c\tau}(Y\rho)), \tag{A.3}$$

thus, the task amounts to evaluation of all correlation functions of the form

$$\langle O_1(t+\tau)O_2(t)\rangle\,;\langle O_1(t+\tau),O_1(t)\rangle\,;\qquad O_1,O_2\in\{a_i,a_i^\dagger,a_i^\dagger a_j\},\qquad i,j=1,2. \tag{A.4}$$

A.2 Quantum Regression Formula

The quantum regression formula states that given a complete set of operators $\mathbf{A}_\mu$, $\mu=1,2,\dots$ such that its expectation values have the property

$$\frac{d}{dt}\langle A_\mu\rangle = \sum_\lambda M_{\mu,\lambda}\langle A_\lambda\rangle \tag{A.5}$$

for some set of constants $M_{\mu,\lambda}$, then we can write down a set of differential equations for the correlations functions $\langle O(t)A_\mu(t+\tau)\rangle$, $\langle A_\mu(t+\tau)O(t)\rangle$, $\mu=1,2,\dots$:

$$\frac{d}{d\tau}\langle O(t)A_\mu(t+\tau)\rangle = \sum_\lambda M_{\mu,\lambda}\langle O(t)A_\lambda(t+\tau)\rangle\,, \tag{A.6}$$

$$\frac{d}{d\tau}\langle A_\mu(t+\tau)O(t)\rangle = \sum_\lambda M_{\mu,\lambda}\langle A_\lambda(t+\tau)O(t)\rangle\,, \tag{A.7}$$

where $O(t)$ can be any system operator. A derivation of this statement is available in [21]. This formula turns out to be exceptionally useful in our situation.

A.3 Evaluation of Correlation Functions

To evaluate the correlation functions, we first calculate

$$\dot{\langle a_1 \rangle} = -\kappa \langle a_1 \rangle \tag{A.8}$$

$$\dot{\left\langle a_1^\dagger \right\rangle} = -\kappa \left\langle a_1^\dagger \right\rangle \tag{A.9}$$

$$\dot{\left\langle a_1^\dagger a_1 \right\rangle} = -2\kappa \left\langle a_1^\dagger a_1 \right\rangle \tag{A.10}$$

$$\dot{\left\langle a_1 a_1^\dagger \right\rangle} = -2\kappa \left\langle a_1^\dagger a_1 \right\rangle \tag{A.11}$$

$$\dot{\langle a_2 \rangle} = -\kappa \langle a_2 \rangle - 2\kappa \langle a_1 \rangle \tag{A.12}$$

$$\dot{\left\langle a_2^\dagger \right\rangle} = -\kappa \left\langle a_2^\dagger \right\rangle - 2\kappa \left\langle a_1^\dagger \right\rangle \tag{A.13}$$

$$\dot{\left\langle a_2^\dagger a_2 \right\rangle} = -2\kappa \left[\left\langle a_2^\dagger a_2 \right\rangle + \left\langle a_1 a_2^\dagger \right\rangle + \left\langle a_1^\dagger a_2 \right\rangle \right] \tag{A.14}$$

$$\dot{\left\langle a_2 a_2^\dagger \right\rangle} = -2\kappa \left[\left\langle a_2^\dagger a_2 \right\rangle + \left\langle a_2^\dagger a_1 \right\rangle + \left\langle a_1^\dagger a_2 \right\rangle \right] \tag{A.15}$$

$$\dot{\left\langle a_1 a_2^\dagger \right\rangle} = -2\kappa \left[\left\langle a_1 a_2^\dagger \right\rangle + \left\langle a_1^\dagger a_1 \right\rangle \right] \tag{A.16}$$

$$\dot{\left\langle a_1^\dagger a_2 \right\rangle} = -2\kappa \left[\left\langle a_1^\dagger a_2 \right\rangle + \left\langle a_1^\dagger a_1 \right\rangle \right]. \tag{A.17}$$

We note that the time evolution of operators in the Hilbert Space of the first cavity is identical to that when considering the cavity alone. This confirms that the model indeed implies the actions of the second system has no effect on that of the first, and thus formally demonstrates unidirectional coupling.

Now the bad cavity limit implies that the steady state of the cavity modes will be the vacuum state. Thus $\langle O a_k \rangle_{ss} = 0$ for any arbitrary operator O. So the only non-zero terms at steady state are

$$\left\langle a_1 a_1^\dagger \right\rangle_{ss} = \left\langle a_2 a_2^\dagger \right\rangle_{ss} = 1. \tag{A.18}$$

Applying the quantum regression formula, we find that the only possible non-zero correlations are:

$$\frac{d}{d\tau} \left\langle a_1(t) a_1^\dagger(t+\tau) \right\rangle = -\kappa \left\langle a_1(t) a_1^\dagger(t+\tau) \right\rangle \tag{A.19}$$

$$\frac{d}{d\tau} \left\langle a_2(t) a_1^\dagger(t+\tau) \right\rangle = -\kappa \left\langle a_2(t) a_1^\dagger(t+\tau) \right\rangle \tag{A.20}$$

$$\frac{d}{d\tau} \left\langle a_1(t) a_2^\dagger(t+\tau) \right\rangle = -\kappa \left\langle a_1(t) a_2^\dagger(t+\tau) \right\rangle - 2\kappa \left\langle a_1(t) a_1^\dagger(t+\tau) \right\rangle \tag{A.21}$$

$$\frac{d}{d\tau} \left\langle a_2(t) a_2^\dagger(t+\tau) \right\rangle = -\kappa \left\langle a_2(t) a_2^\dagger(t+\tau) \right\rangle - 2\kappa \left\langle a_2(t) a_1^\dagger(t+\tau) \right\rangle \tag{A.22}$$

$$\frac{d}{d\tau}\left\langle a_1(t+\tau)a_1^\dagger(t)\right\rangle = -\kappa\left\langle a_1(t+\tau)a_1^\dagger(t)\right\rangle \tag{A.23}$$

$$\frac{d}{d\tau}\left\langle a_2(t+\tau)a_1^\dagger(t)\right\rangle = -\kappa\left\langle a_2(t+\tau)a_1^\dagger(t)\right\rangle - 2\kappa\left\langle a_1(t+\tau)a_1^\dagger(t)\right\rangle \tag{A.24}$$

$$\frac{d}{d\tau}\left\langle a_1(t+\tau)a_2^\dagger(t)\right\rangle = -\kappa\left\langle a_1(t+\tau)a_2^\dagger(t)\right\rangle \tag{A.25}$$

$$\frac{d}{d\tau}\left\langle a_2(t+\tau)a_2^\dagger(t)\right\rangle = -\kappa\left\langle a_2(t+\tau))a_2^\dagger(t)\right\rangle - 2\kappa\left\langle a_1(t+\tau)a_2^\dagger(t)\right\rangle \tag{A.26}$$

$$\frac{d}{d\tau}\left\langle a_2(t+\tau)a_1^\dagger(t)\right\rangle = -\kappa\left\langle a_2(t))a_1^\dagger(t)\right\rangle - 2\kappa\left\langle a_1(t+\tau)a_1^\dagger(t)\right\rangle \tag{A.27}$$

Solving these equations, and setting $t \to \infty$. We see that

$$\left\langle a_1(t)a_1^\dagger(t+\tau)\right\rangle_{ss} = \left\langle a_2(t)a_2^\dagger(t+\tau)\right\rangle_{ss} = e^{-\kappa t} \tag{A.28}$$

$$\left\langle a_1(t)a_2^\dagger(t+\tau)\right\rangle_{ss} = \left\langle a_1(t)a_2^\dagger(t)\right\rangle_{ss} e^{-\kappa\tau} - 2\kappa\tau e^{-\kappa\tau} = -2\kappa\tau e^{-\kappa\tau} \tag{A.29}$$

$$\left\langle a_2(t+\tau)a_1^\dagger(t)\right\rangle_{ss} = \left\langle a_2(t)a_1^\dagger(t)\right\rangle_{ss} e^{-\kappa\tau} - 2\kappa\tau e^{-\kappa\tau} = -2\kappa\tau e^{-\kappa\tau} \tag{A.30}$$

are the only non-zero correlations at steady state.

A.4 Evaluation of Required Traces

Using the above results, we can evaluate the only non-zero terms

$$\begin{aligned}
\int_0^\infty d\tau \mathrm{Tr}\left[a_1 e^{\mathcal{L}_c\tau}(a_1^\dagger\rho_c^{ss})\right] &= \int_0^\infty d\tau \mathrm{Tr}\left[a_2 e^{\mathcal{L}_c\tau}(a_2^\dagger\rho_c^{ss})\right] = \int_0^\infty e^{-\kappa\tau}d\tau = \frac{1}{\kappa} \\
\int_0^\infty d\tau \mathrm{Tr}\left[a_2 e^{\mathcal{L}_c\tau}(a_1^\dagger\rho_c^{ss})\right] &= \int_0^\infty d\tau \mathrm{Tr}\left[a_2 e^{\mathcal{L}_c\tau}(a_1^\dagger\rho_c^{ss})\right] = \int_0^\infty -2\kappa e^{-\kappa\tau}d\tau = -\frac{2}{\kappa}
\end{aligned}$$

which coincides with the statement given in (4.57).

Appendix B

Continuous Variable Quantum States

In this section, we outline the essential aspects of the theory of continuous quantum variables. Such variables are frequently used to describe the modes of a quantized electromagnetic field, but also arise in many other situations, such as the quantized motional states of a harmonic oscillator.

As the thesis is primarily concerned with entanglement, a physical resource that is independent of its physical implementation, we will not need to be too concerned with exactly what the continuous state represents. Thus, for the sake of simplicity, we will use the familiar notation of quantum optics. Since this is not the primary focus of the thesis, this exposition will be very concise. The interested reader is encouraged to refer to textbooks, such as Walls and Milburn [51], for further details.

B.1 Fundamentals

A state with a definite number of bosons n is represented by $|n\rangle$, which, more generally, is an eigenstate of the energy operator. Creation and annihilation operators are then defined by

$$a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{B.1}$$

respectively, with commutation relation $[a, a^\dagger] = 1$.

B.2 Quadrature Operators

The generalized quadrature operator is defined by

$$X(\theta) = a^\dagger e^{i\theta} + a e^{-i\theta}, \tag{B.2}$$

from which we can define two special quadrature operators that correspond to specific values of θ, i.e:

$$X = a + a^\dagger \qquad Y = -i(a - a^\dagger) \tag{B.3}$$

which have the commutation relation $[X, Y] = 2i$ that result in the Heisenberg uncertainty relation

$$\Delta X \Delta Y \geq 1 \tag{B.4}$$

where $(\Delta X)^2 = \langle X^2 \rangle - \langle X \rangle^2$, $(\Delta Y)^2 = \langle Y^2 \rangle - \langle Y \rangle^2$ are the quadrature uncertainties.

B.3 Minimum Uncertainty States

A minimum uncertainty state is a state that satisfies

$$\Delta X \Delta Y = 1. \tag{B.5}$$

In general, such states are defined by the equation

$$|\alpha, \epsilon\rangle = D(\alpha)S(\epsilon)|0\rangle \tag{B.6}$$

where

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a), \tag{B.7}$$

$$S(\epsilon) = \exp(\frac{1}{2}\epsilon^* a^2 - \frac{1}{2}\epsilon a^{\dagger 2}), \tag{B.8}$$

are the displacement and squeezing operators respectively.

B.3.1 Squeezed States

Physically, the squeezing operator reduces the uncertainty along one fixed quadrature $X(\theta)$ at the cost of introducing extra uncertainty in the direction orthogonal to $X(\theta)$ (i.e $X(\theta + \frac{\pi}{2})$. That is, given a minimum uncertainty state $|\alpha, re^{i\phi}\rangle$, then

$$\Delta X\left(\frac{\phi}{2}\right) = e^{-r}, \qquad \Delta X\left(\frac{1}{2}(\phi + \pi)\right) = e^{r}. \tag{B.9}$$

The squeezing operator has the nice property that

$$S^{-1}(\epsilon) = S(-\epsilon), \tag{B.10}$$

which is used to show in Chapter 6 that the steady state of the cascaded qubit system in the many atom limit is simply the product of two squeezed states.

B.3.2 Coherent States

If a state is not squeezed, then $S(\epsilon)$ becomes a phase factor that can be neglected. The resulting state $|\alpha, 0\rangle$ is defined as a *Coherent State*. Coherent states can be expanded in the form

$$|\alpha, 0\rangle = e^{-|\alpha|^2} \sum_n^\infty \frac{\alpha^n}{\sqrt{n}} |n\rangle, \tag{B.11}$$

and are also eigenstates of the annihilation operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{B.12}$$

B.4 Two-mode Squeezed States

Entanglement can only be defined in a system with two or more subsystems. Thus, for us to characterize entanglement, there needs to be at least two modes. The simplest example of a pair of entangled continuous states is the two-mode squeezed state, defined by

$$S(\epsilon)|0\rangle_1|0\rangle_2 = \exp(\epsilon^* a_1 a_2 - \epsilon a_2^\dagger a_1^\dagger)|0\rangle_1|0\rangle_2, \tag{B.13}$$

where a_1 and a_2 are the annihilation operators of the first and second mode respectively. A two-mode squeezed state can be characterized by the two mode quadratures

$$X^+ = X_1 \pm X_2, \qquad Y^+ = Y_1 \pm Y_2. \tag{B.14}$$

For example, suppose ϵ is real, then we have quadrature uncertainties

$$\left\langle (X_1 \pm X_2)^2 \right\rangle = e^{\pm 2\epsilon}, \qquad \left\langle (Y_1 \pm Y_2)^2 \right\rangle = e^{\mp 2\epsilon} \tag{B.15}$$

In particular, a state is two mode squeezed provided $|\epsilon| \neq 1$. In Chapter 2, we will see that this is also an indication of entanglement between the two bosonic modes.

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