Eigenvalues of Driven Jaynes Cummings System

Jonathan Mackrory
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#### Abstract

The Jaynes Cummings Model with driving is considered. We attempt to extend previous results to include atomic and cavity detunings. Analytical expressions for the energies are derived for the case of atomic detuning. For the case of cavity detuning, the solution scheme fails and no exact solutions are possible. Non-degenerate perturbation theory is used to find the first order energy shifts for the case of small detuning. These analytical results are compared with fully numerical simulations of the system with detunings.


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## Chapter 1

## Introduction

The interaction of matter with radiation has been one of the driving forces of modern physics. The problem of blackbody radiation led Planck to reluctantly introduce the idea of quantisation at the turn of the previous century. In describing the photoelectric effect Einstein, in one of his seminal papers of 1905, introduced the concept of the photon. The problems of spectral radiation from atoms culminated in the development of Quantum Mechanics during the 1920s. Physics' most accurate theory, Quantum Electrodynamics(QED) describes the interaction of electrons and other leptons with the electromagnetic field. The advent of the laser has not only opened up new vistas in physics research, but also revolutionised communications.

In more recent years the advent of atom trapping and optical microcavities has opened new opportunities to test fundamental physics and possible links to future technologies, for example Quantum Computing. One of the areas of Quantum Optics that relates to this is known as Cavity Quantum Electrodynamics(CQED). CQED is an area of considerable theoretical and experimental interest. The core system of Cavity QED is a that of cold atoms held inside an optical resonator interacting with external laser light fields. In this report we examine a model of this system.

The Jaynes Cummings Model describes a two level atom in the radiation field. Much of the initial work on two level systems was undertaken in the context of magnetic resonance of spin- $1 / 2$ particles. In an optics context this two level model was put forward in 1963 by Jaynes and Cummings[1].

The Jaynes Cummings Model is the one of the simplest systems in quantum optics. Not only can it be solved exactly but it also displays interesting phenomena of more general interest, e.g. collapse-revival phenomena. This means that the Jaynes-Cummings Model serves as a useful approximation to more complicated systems. Hence it is still a research topic some 40 years after its introduction.

This work examines an extension of the Jaynes Cummings Model that includes an external driving field. This introduction of the driving field is necessary to provide comparison between theory and experiment, as in experiments the atom-cavity system is probed by an external laser field to retrieve information about the system. We begin with a section outlining the Jaynes Cummings Model and its solutions. The displacement and squeezing transformations that are required are then introduced. Proceeding onto the body of the work we cover
a minor extension of analytical work undertaken by Alsing and coworkers[2]. First order perturbation theory is used to derive results in regimes where the exact methods fail. Finally we look at some numerical simulations that compliment work of earlier sections.

## Chapter 2

## Background

This section covers the necessary background to understanding the body of this report. We begin with a discussion of the Jaynes Cummings Model and its solutions. This is followed by sections discussing the Displacement and Squeezing Operators. The information contained herein relies upon the textbooks of Walls[3], and Gerry and Knight[4].

### 2.1 Jaynes Cummings Model

The Jaynes Cummings Model is an idealised model of the interaction of an atom with the radiation field. We assume that we can isolate and drive one particular one atomic transition and hence ignore the rest of the atomic structure. So we can model the atom as having only two levels. For simplicity we only consider the interaction with a single mode of radiation. The interaction of radiation and the atom is described by the dipole interaction, $\hat{H}_{I}=-\mathbf{d} \cdot \mathbf{E}$ where $\mathbf{d}$ and $\mathbf{E}$ are the atomic dipole moment and the Electric field respectively. Upon imposing quantisation this becomes

$$
i \hbar g\left(a^{\dagger}-a\right)\left(\sigma_{-}+\sigma_{+}\right)
$$

Here $g$ is the dipole coupling constant that takes up the normalisation constant from the EM wave as well as the dipole matrix elements. $a$ and $a^{\dagger}$ are the creation and annihilation operators for the radiation field. They obey the usual commutation relations

$$
\begin{equation*}
\left[a, a^{\dagger}\right]=1 \tag{2.1}
\end{equation*}
$$

The atomic raising and lowering operators $\sigma_{ \pm}$, and atomic inversion operator $\sigma_{z}$ are defined as
where $|+\rangle$ is the excited state and $|-\rangle$ is the ground state. From the definitions it is easily shown that these operators are formally identical to the Pauli spin
operators and hence satisfy the commutation relations

$$
\begin{align*}
{\left[\sigma_{+}, \sigma_{-}\right] } & =\sigma_{z}  \tag{2.5}\\
{\left[\sigma_{z}, \sigma_{ \pm}\right] } & = \pm 2 \sigma_{ \pm} \tag{2.6}
\end{align*}
$$

It follows that we may express the Hamiltonian of the bare atom as

$$
\hat{H}_{A}=\hbar \omega_{a} / 2(|+\rangle\langle+|-|-\rangle\langle-|)=\hbar \omega_{a} / 2 \sigma_{z}
$$

using the fact that we have set the zero of our energy scale halfway between the ground and excited state energies.

We may drop the non energy conserving terms like $a^{\dagger} \sigma_{+}$and $a \sigma_{-}$which respectively create a photon while exciting the atom and annihilate a photon while the atom decays to its lower state. Finally the Jaynes Cummings Hamiltonian is given by

$$
\begin{equation*}
H=\hbar \omega_{c} a^{\dagger} a+\frac{\hbar \omega_{a}}{2} \sigma_{z}+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right) \tag{2.7}
\end{equation*}
$$

The first term corresponds to the energy of the photons in the cavity, where $\omega_{c}$ is the resonant frequency of the cavity. The second term to the atomic energy, where $\omega_{a}$ is the frequency of the atomic transition that we have isolated. The third term describes the interaction between the two via electric dipole transitions. This may be solved exactly by noting that the interaction term couples pairs of states, $|n,+\rangle$ and $|n+1,-\rangle$. I will omit the details of the calculation, and merely present the final results. These are taken from Gerry and Knight[4].

$$
\begin{align*}
E_{n, \xi} & =\hbar \omega_{c}\left(n+\frac{1}{2}\right) \pm \hbar \Omega_{n}(\Delta)  \tag{2.8}\\
\left|\psi_{n, u}\right\rangle & =\frac{1}{\sqrt{2}}\left(\frac{\Omega_{n}(\Delta)+\Delta}{\Omega_{n}(\Delta)}|n,+\rangle+\frac{\Omega_{n}(\Delta)-\Delta}{\Omega_{n}(\Delta)}|n+1,-\rangle\right)  \tag{2.9}\\
\left|\psi_{n, l}\right\rangle & =\frac{1}{\sqrt{2}}\left(\frac{\Omega_{n}(\Delta)-\Delta}{\Omega_{n}(\Delta)}|n,+\rangle+\frac{\Omega_{n}(\Delta)+\Delta}{\Omega_{n}(\Delta)}|n+1,-\rangle\right)  \tag{2.10}\\
\Omega_{n}(\Delta) & =\left[\frac{\left(\omega_{a}-\omega_{c}\right)^{2}}{4}+g^{2}(n+1)\right]^{1 / 2} \Delta=\omega_{a}-\omega_{c} \tag{2.11}
\end{align*}
$$

### 2.2 Displacement Operator

The Displacement operator is used to generate coherent states. Coherent states are of interest as they are as close as a quantum mechanical state comes to a classical wave of well defined amplitude and phase. As a result coherent states and the displacement operator are ubiquitous in Quantum Optics.

A coherent state is created by applying the displacement operator to the ground state of the harmonic oscillator. The displacement operator is defined as

$$
\begin{equation*}
D(\alpha)=\exp \left(\alpha a^{\dagger}-\alpha^{*} a\right) \tag{2.12}
\end{equation*}
$$

It is a unitary operator and hence obeys

$$
\begin{equation*}
D^{\dagger}(\alpha)=D^{-1}(\alpha)=D(-\alpha) \tag{2.13}
\end{equation*}
$$

The quantised radiation field is a set of quantised harmonic oscillators, so the vacuum state, denoted $|0\rangle$, corresponds to the ground state of a harmonic oscillator. A coherent state of complex amplitude $\alpha$ is given by

$$
\begin{equation*}
|\alpha\rangle=D(\alpha)|0\rangle \tag{2.14}
\end{equation*}
$$

The displacement operator can be applied to any number state, displacing it away from the origin. Unlike the displaced vacuum, these states seem to lack any larger meaning.

Throughout this report we will need to apply unitary transformations. Of considerable use in calculating these is the Baker-Hausdorff Lemma [5]

$$
\begin{equation*}
e^{i \lambda B} A e^{-i \lambda B}=A+i \lambda[B, A]+\frac{(i \lambda)^{2}}{2!}[B,[B, A]]+\ldots \tag{2.15}
\end{equation*}
$$

where $B$ is a hermitian operator and $\lambda$ is real. Applying this we can transform the creation and annihilation operators according to

$$
\begin{align*}
D^{\dagger}(\alpha) a D(\alpha) & =a+\alpha  \tag{2.16}\\
D^{\dagger}(\alpha) a^{\dagger} D(\alpha) & =a^{\dagger}+\alpha^{*} \tag{2.17}
\end{align*}
$$

As suggested by the name, this operator displaces the vacuum state away from the origin. Under time evolution the wavefunction oscillates about the origin at resonant frequency $\omega$ in a similar fashion to that of a classical harmonic oscillator. This can be made explicit by transforming to the Heisenberg picture where the operators carry all the time dependence. We introduce the unitary time evolution operator

$$
\begin{align*}
\hat{U}(t) & =\exp [-i \hat{H} t / \hbar]  \tag{2.18}\\
& =\exp \left[-i \omega a^{\dagger} a t\right]
\end{align*}
$$

The second line specialising to the case of the harmonic oscillator Hamiltonian $H=\hbar \omega a^{\dagger} a$ as our state is a harmonic oscillator eigenstate. In the Heisenberg picture the annihilation operator is given by

$$
\begin{equation*}
a(t)=U^{\dagger}(t) a U(t)=a(0) e^{-i \omega t} \tag{2.19}
\end{equation*}
$$

We can apply a similar transformation to the the displacement operator.

$$
\begin{align*}
U^{\dagger}(t) D(\alpha) U(t) & =U^{\dagger} \sum_{n=0}^{\infty} \frac{\left[\alpha a^{\dagger}-\alpha^{*} a\right]^{n}}{n!} U \\
& =\sum_{n=0}^{\infty} \frac{\left[\alpha U^{\dagger} a^{\dagger} U-\alpha^{*} U^{\dagger} a U\right]^{n}}{n!} \\
& =\exp \left[\alpha e^{i \omega t} a^{\dagger}-\alpha^{*} e^{-i \omega t} a\right] \tag{2.20}
\end{align*}
$$

where in the second line we have used the unitarity of $U(t)$ to insert $U^{\dagger} U=1$ as required. It follows that under time evolution a coherent state evolves as

$$
|\alpha(t)\rangle=U(t)|\alpha\rangle=\left|\alpha e^{i \omega t}\right\rangle
$$

In comparison with Eq.(2.16) we see that now the operators under time evolution transform according to

$$
\begin{align*}
D^{\dagger}\left(\alpha e^{-i \omega t}\right) a D\left(\alpha e^{-i \omega t}\right) & =a+\alpha e^{-i \omega t}  \tag{2.21}\\
D^{\dagger}\left(\alpha e^{-i \omega t}\right) a^{\dagger} D\left(\alpha e^{-i \omega t}\right) & =a+\alpha^{*} e^{i \omega t} \tag{2.22}
\end{align*}
$$

So we see that the displacement of the state oscillates sinusoidally with time, confirming the earlier comparison to the motion of a classical harmonic oscillator.

### 2.3 Squeezing Operator

It is a general result of quantum mechanics that two incompatible observables must satisfy an uncertainty principle, the product of their variances has a nonzero lower bound. The most quoted example is the well known positionmomentum uncertainty relation

$$
\Delta x \Delta p \geq \frac{\hbar}{2}
$$

This can be recast in a dimensionless form which is useful for optics. The two quadratures are $\hat{X}_{1}, \hat{X}_{2}$, analogous to the canonical position, $\hat{x}$, and momentum $\hat{p}$. The quadrature operators defined as

$$
\begin{align*}
& \hat{X}_{1}=\frac{a+a^{\dagger}}{2}=\sqrt{\frac{m \omega}{2 \hbar}} \hat{x}  \tag{2.23}\\
& \hat{X}_{2}=\frac{a-a^{\dagger}}{2 i}=\frac{1}{\sqrt{2 \hbar m \omega}} \hat{p} \tag{2.24}
\end{align*}
$$

where we have used the definitions of $a, a^{\dagger}$

$$
a=\sqrt{\frac{m \omega}{2 \hbar}}\left(\hat{x}+i \frac{\hat{p}}{m \omega}\right) \quad a^{\dagger}=\sqrt{\frac{m \omega}{2 \hbar}}\left(\hat{x}+i \frac{\hat{p}}{m \omega}\right)
$$

The quadrature operators must satisfy the following uncertainty relation.

$$
\begin{equation*}
\Delta X_{1} \Delta X_{2} \geq 1 / 2 \tag{2.25}
\end{equation*}
$$

In the vacuum state the equality is satisfied with equal uncertainty in both quadratures. The squeezing transformation reduces the quantum uncertainty of a state in one quadrature at the expense of increased uncertainty in the other quadrature. This has application in increasing the accuracy of physical measurements, whereby we can reduce the quantum uncertainty on our measurement channel below vacuum levels.

The squeezing operator is defined as

$$
\begin{equation*}
S(\eta)=\exp \left[\frac{1}{2}\left(\eta a^{\dagger^{2}}-\eta^{*} a^{2}\right)\right] \tag{2.26}
\end{equation*}
$$

where the squeeze parameter $\eta$ is complex. Like the Displacement operator, $S(\eta)$ is unitary, so that

$$
S^{\dagger}(\eta)=S^{-1}(\eta)=S(-\eta)
$$

Using the Baker-Hausdorff Lemma it can be shown that the creation and annihilation operators transform according to

$$
\begin{align*}
S^{\dagger}(\eta) a S(\eta) & =a \cosh r-a^{\dagger} e^{i \theta} \sinh r  \tag{2.27}\\
S^{\dagger}(\eta) a^{\dagger} S(\eta) & =a^{\dagger} \cosh r-a e^{-i \theta} \sinh r \tag{2.28}
\end{align*}
$$

where we have used $\eta=r e^{i \theta}$. The effect of $r$ is to set the magnitude of the squeezing, $\theta$ determines the rotation of the uncertainty ellipse with respect to the quadrature axes.

For example if we consider the squeezed vacuum state

$$
\begin{equation*}
|\eta\rangle=S^{\dagger}(\eta)|0\rangle, \tag{2.29}
\end{equation*}
$$

then we can calculate the variance of the quadrature operators with respect to the squeezed state.

$$
\begin{align*}
& \left\langle\left(\Delta \hat{X}_{1}\right)^{2}\right\rangle=\frac{1}{4}\left[\sinh ^{2} r+\cosh ^{2} r-2 \cos \theta \cosh r \sinh r\right]  \tag{2.30}\\
& \left\langle\left(\Delta \hat{X}_{2}\right)^{2}\right\rangle=\frac{1}{4}\left[\sinh ^{2} r+\cosh ^{2} r+2 \cos \theta \cosh r \sinh r\right] . \tag{2.31}
\end{align*}
$$

Then for $\theta=0$

$$
\begin{align*}
& \left\langle\left(\Delta \hat{X}_{1}\right)^{2}\right\rangle=\frac{1}{4} e^{-2 r}  \tag{2.32}\\
& \left\langle\left(\Delta \hat{X}_{2}\right)^{2}\right\rangle=\frac{1}{4} e^{2 r} \tag{2.33}
\end{align*}
$$

So we see that in this instance the $X_{1}$ quadrature is squeezed.
In a similar fashion to that of the displacement operator we can introduce time evolution with the results that

$$
\begin{equation*}
U^{\dagger} S(\eta) U=\exp \left[\frac{1}{2}\left(\eta e^{2 i \omega t} a^{\dagger^{2}}-\eta^{*} e^{-2 i \omega t} a^{2}\right)\right] \tag{2.34}
\end{equation*}
$$

Having discussed the properties of the squeezing operator I should at least mention the states. A squeezed vacuum state is created via

$$
\begin{equation*}
|\eta\rangle=S(\eta)|0\rangle \tag{2.35}
\end{equation*}
$$

As with the coherent states it is possible to apply the squeezing transformation to any state. A common extension it to combine displacement and squeezing to create a displaced squeezed number state

$$
|\alpha, \eta ; n\rangle=D(\alpha) S(\eta)|n\rangle
$$

## Chapter 3

## Analytical Work

This chapter covers the work undertaken in carrying out the extension of Alsing's results[2]. The solution scheme is identical and the final results are even contained as an aside in the original paper. Despite this, this has been what I have spent the largest share of my time working on, in addition to being the vehicle for learning more about Quantum Optics. The debt owed to the original work cannot be overstated. In this section we look at the extension of the Jaynes Cummings Model to include an external laser driving field. This can be oriented to drive the atom-cavity system by either directly driving the atoms through the side of the cavity or by driving the cavity by coupling through one of the mirrors.

The Jaynes Cummings Model with a classical driving field is given by the following Hamiltonian
$H=\hbar \omega_{c} a^{\dagger} a+\frac{\hbar \omega_{a}}{2} \sigma_{z}+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right)+i \hbar \mathcal{E}\left[\binom{\sigma_{+}}{a^{\dagger}} e^{-i \omega_{L} t}-\binom{\sigma_{-}}{a} e^{i \omega_{L} t}\right]$
In the final term, the top line applies to the laser driving the atom, and the lower line for the laser driving the cavity. The laser is at frequency $\omega_{L}$.

We can make a unitary transformation to an "interaction" picture which eliminates the time dependence of the driving field. The "interaction" picture used does not follow the usual scheme of eliminating the bare Hamiltonian, hence the quotation marks. This transformation is required in order to allow any progress in finding the energy eigenvalues and eigenstates of the system. The energy eigenvalues that we will find are better termed quasi-energies as they define shifts in the energy levels around the energies of the Hamiltonian used in the unitary transformation. These energy shifts are added to every energy level of the transforming Hamiltonian $E_{m}=\hbar \omega_{L}(m-1 / 2)$ for $m=0,1,2, \ldots$.

In this case we apply the following unitary transformation

$$
\begin{equation*}
U(t)=\exp \left(\frac{-i}{\hbar}\left[\hbar \omega_{L} a^{\dagger} a+\frac{\hbar \omega_{L}}{2} \sigma_{z}\right] t\right) \tag{3.2}
\end{equation*}
$$

The operators and states transform according to

$$
\hat{A}(t)=U^{\dagger}(t) \hat{A} U(t)
$$

$$
\left|\psi^{I}\right\rangle=U^{\dagger}(t)\left|\psi^{S}\right\rangle
$$

From the Baker-Hausdorff Theorem, we find that

$$
\begin{align*}
a(t) & =a(0) e^{-i \omega_{L} t}  \tag{3.3}\\
\sigma_{-}(t) & =\sigma_{-}(0) e^{-i \omega_{L} t} \tag{3.4}
\end{align*}
$$

with the adjoint expressions following from these. In the "interaction"picture we have the following problem

$$
\begin{equation*}
i \hbar \frac{d\left|\psi^{I}\right\rangle}{d t}=\left[\hbar \Delta_{c} a^{\dagger} a+\frac{\hbar \Delta_{a}}{2} \sigma_{z}+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right)+i \hbar \mathcal{E}\binom{\sigma_{+}-\sigma_{-}}{a^{\dagger}-a}\right]\left|\psi^{I}\right\rangle \tag{3.5}
\end{equation*}
$$

where we have defined

$$
\begin{equation*}
\Delta_{c}=\omega_{c}-\omega_{L} \quad \Delta_{a}=\omega_{a}-\omega_{L} \tag{3.6}
\end{equation*}
$$

Following normal procedure we may now seek solutions to this equation that satisfy

$$
\begin{equation*}
H\left|\psi^{I}\right\rangle=E\left|\psi^{I}\right\rangle \tag{3.7}
\end{equation*}
$$

We will proceed by deriving the results for atomic detuning i.e. $\omega_{a} \neq \omega_{c}=$ $\omega_{L}$. The solution scheme we follow is borrowed from [2]. I will then show how this method fails for the case of a cavity detuning where $\omega_{c} \neq \omega_{L}$. From here the need for numerical simulations becomes clear.

### 3.1 Atomic Detuning, Atomic Driving

In this section we look at the case where the laser is coupling to the atoms and is on resonance with the cavity, but $\Delta_{a} \neq 0$. In this case the eigenvalue problem is

$$
\begin{equation*}
\left[\frac{\hbar \Delta_{a}}{2} \sigma_{z}+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right)+i \hbar \mathcal{E}\left(\sigma_{+}-\sigma_{-}\right)\right]\left|\psi_{E}\right\rangle=E\left|\psi_{E}\right\rangle \tag{3.8}
\end{equation*}
$$

We assume a solution of the form

$$
\begin{equation*}
|\psi\rangle=\left|\psi_{E}^{+}\right\rangle|+\rangle+\left|\psi_{E}^{-}\right\rangle|-\rangle \tag{3.9}
\end{equation*}
$$

On substituting this in and projecting against $\langle \pm|$ we find the coupled equations

$$
\begin{align*}
\left(E-\frac{\hbar \Delta_{a}}{2}\right)\left|\psi_{E}^{+}\right\rangle+(i \hbar g a-i \hbar \mathcal{E})\left|\psi_{E}^{-}\right\rangle & =0  \tag{3.10}\\
\left(-i \hbar g a^{\dagger}+i \hbar \mathcal{E}\right)\left|\psi_{E}^{+}\right\rangle+\left(E+\frac{\hbar \Delta_{a}}{2}\right)\left|\psi_{E}^{-}\right\rangle & =0 \tag{3.11}
\end{align*}
$$

From Eq.(3.10) we have

$$
\begin{equation*}
\left|\psi_{E}^{+}\right\rangle=\frac{-(i \hbar g a-i \hbar \mathcal{E})}{E-\hbar \Delta_{a} / 2}\left|\psi_{E}^{-}\right\rangle \tag{3.12}
\end{equation*}
$$

On substituting into Eq.(3.11) we obtain

$$
\begin{equation*}
\left(E^{2}-\frac{\hbar^{2} \Delta_{a}^{2}}{4}\right)\left|\psi_{E}^{-}\right\rangle=\hbar^{2} g^{2}\left(a^{\dagger}-\mathcal{E} / g\right)(a-\mathcal{E} / g)\left|\psi_{E}^{-}\right\rangle \tag{3.13}
\end{equation*}
$$

The above equation may be transformed to the harmonic oscillator eigenvalue equation by applying the unitary displacement operator. If we multiply Eq.(3.13) by $D^{\dagger}(\mathcal{E} / g)$ and insert $D^{\dagger}(\mathcal{E} / g) D(\mathcal{E} / g)=1$ as required then we arrive at the transformed equation

$$
\begin{equation*}
a^{\dagger} a\left|\tilde{\psi}_{E}^{-}\right\rangle=\left(\frac{E^{2}}{\hbar^{2} g^{2}}-\frac{\Delta_{a}^{2}}{4 g^{2}}\right)\left|\tilde{\psi}_{E}^{-}\right\rangle \tag{3.14}
\end{equation*}
$$

where

$$
\begin{equation*}
\left|\tilde{\psi}_{E}^{-}\right\rangle=D^{\dagger}(\mathcal{E} / g)\left|\psi_{E}^{-}\right\rangle \tag{3.15}
\end{equation*}
$$

Evidently Eq.(3.14) is satisfied by the number states. Since we must have

$$
a^{\dagger} a|n\rangle=n|n\rangle,
$$

the energies must satisfy

$$
\begin{equation*}
\left(\frac{E^{2}}{\hbar^{2} g^{2}}-\frac{\Delta_{a}^{2}}{4 g^{2}}\right)=n \tag{3.16}
\end{equation*}
$$

It follows that

$$
\begin{align*}
E_{0} & =\frac{\hbar \Delta_{a}}{2}  \tag{3.17}\\
E_{n, u} & =+\hbar g \sqrt{n+\frac{\Delta_{a}^{2}}{4 g^{2}}}  \tag{3.18}\\
E_{n, l} & =-\hbar g \sqrt{n+\frac{\Delta_{a}^{2}}{4 g^{2}}} \tag{3.19}
\end{align*}
$$

Upon inverting the displacement of the eigenstates and using Eq.(3.12) we have

$$
\begin{gather*}
\left|\psi_{E}^{-}\right\rangle \equiv\left\{\begin{array}{r}
\left|\psi_{0}^{-}\right\rangle=|\mathcal{E} / g, 0\rangle \\
\left|\psi_{n, \xi}^{-}\right\rangle=|\mathcal{E} / g, n\rangle
\end{array} \quad n=1,2, \ldots \xi=u, l\right.  \tag{3.20}\\
\left|\psi_{E}^{+}\right\rangle \equiv\left\{\begin{array}{l}
\left|\psi_{0}^{-}\right\rangle=0 \\
\left|\psi_{n, \xi}^{-}\right\rangle=\frac{-i \hbar g \sqrt{n}}{E_{n, \xi}-\hbar \Delta_{a} / 2}|\mathcal{E} / g, n-1\rangle \quad n=1,2, \ldots
\end{array}\right. \tag{3.21}
\end{gather*}
$$

Corresponding to Eq.(3.17) we have the state

$$
\begin{equation*}
\left|\psi_{0}\right\rangle=|\mathcal{E} / g, 0\rangle|-\rangle \tag{3.22}
\end{equation*}
$$

The other states are specified by
$\left|\psi_{n, \xi}\right\rangle=c_{n, \xi}\left(-i \hbar \sqrt{n}|\mathcal{E} / g, n-1\rangle|+\rangle+\left(E_{n, \xi}-\hbar \Delta_{a} / 2\right)|\mathcal{E} / g, n\rangle|-\rangle\right) n=1,2, \ldots$
The above expression is not normalised as the full expression is needlessly messy without adding any insight. Nonetheless we may note that in addition to recovering the resonant states as $\Delta_{a} \rightarrow 0$ and $g \gg \Delta_{a}$, the excited and ground
state of the atom effectively decouple as $\Delta_{a} \rightarrow \infty$. On the approach to the limit $\Delta_{a}=\infty$ there is a small dispersive interaction.

We now return to the Schrödinger picture by inverting the unitary transformation carried out in Eq.(3.5). We find the following expressions for the states

$$
\begin{equation*}
\left|\psi_{0}^{S}\right\rangle=e^{-\omega_{L} t / 2}\left|e^{-i \omega_{L} t} \mathcal{E} / g ;\right\rangle|-\rangle \tag{3.24}
\end{equation*}
$$

and

$$
\begin{gather*}
\left|\psi_{n, \xi}^{S}\right\rangle=c_{n, \xi} e^{-i(n-1 / 2) \omega_{L} t}\left(-i \hbar \sqrt{n}\left|e^{-i \omega_{L} t} \mathcal{E} / g, n-1\right\rangle|+\rangle\right. \\
\left.+\left(E_{n, \xi}-\hbar \Delta_{a} / 2\right)\left|e^{-i \omega_{L} t} \mathcal{E} / g, n\right\rangle|-\rangle\right) \\
n=1,2, \ldots \quad \xi=u, l \tag{3.25}
\end{gather*}
$$

The quasienergies and states have the correct limit as $\Delta_{a} \rightarrow 0$. In the Schrödinger picture there is no simple expression for the time evolution of these states. This is due to the presence of the displacement each state includes components of all number states, and hence all possible harmonics $\omega_{m}=(m-$ $1 / 2) \omega_{L}$ for $m=0,1,2, \ldots$. The quasienergies then define shifts that are applied to this ladder of energies.

### 3.2 Atomic Detuning, Cavity Driving

In this section we move on to the case where the external laser field is driving the cavity. We are still in the regime where $\omega_{a} \neq \omega_{c}=\omega_{L}$. We will proceed by obtaining a set of coupled equations for $\left|\psi_{E}^{+}\right\rangle$and $\left|\psi_{E}^{-}\right\rangle$. These can then be decoupled by judicious use of creation and annihilation operators and their commutation relations. The resulting eigenvalue problems for $\left|\psi_{E}^{-}\right\rangle$is quadratic in $a$ and $a^{\dagger}$. Fortunately this may be transformed into a harmonic oscillator Hamiltonian by a combination of squeezing and displacement transformations. In this case the equation we seek to solve is:

$$
\begin{equation*}
\left[\frac{\hbar \Delta_{a}}{2} \sigma_{z}+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right)+i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right]\left|\psi_{E}\right\rangle=E\left|\psi_{E}\right\rangle \tag{3.26}
\end{equation*}
$$

Again assuming a solution of the form in Eq.(3.9), upon substitution in we obtain the following coupled equations

$$
\begin{align*}
& {\left[\left(E-\frac{\hbar \Delta_{a}}{2}\right)-i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right]\left|\psi_{E}^{+}\right\rangle-i \hbar g a\left|\psi_{E}^{-}\right\rangle=0}  \tag{3.27}\\
& {\left[\left(E+\frac{\hbar \Delta_{a}}{2}\right)-i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right]\left|\psi_{E}^{-}\right\rangle+i \hbar g a^{\dagger}\left|\psi_{E}^{+}\right\rangle=0} \tag{3.28}
\end{align*}
$$

We multiply these two equations from the left by $a^{\dagger}$ and $a$ respectively, and use the commutation relations in Eq.(2.1) to obtain

$$
\begin{aligned}
& {\left[\left(E-\frac{\hbar \Delta_{a}}{2}\right)-i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right] a^{\dagger}\left|\psi_{E}^{+}\right\rangle+i \hbar g a^{\dagger} a\left|\psi_{E}^{-}\right\rangle-i \hbar \mathcal{E}\left|\psi_{E}^{+}\right\rangle=0(3.29)} \\
& {\left[\left(E+\frac{\hbar \Delta_{a}}{2}\right)-i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right] a\left|\psi_{E}^{-}\right\rangle-i \hbar g a a^{\dagger}\left|\psi_{E}^{+}\right\rangle-i \hbar \mathcal{E}\left|\psi_{E}^{-}\right\rangle=0(3.30)}
\end{aligned}
$$

We can solve Eqs.(3.27) for $a\left|\psi_{E}^{+}\right\rangle$and $a^{\dagger}\left|\psi_{E}^{-}\right\rangle$, and substitute these into Eq.(3.29). I will note in passing that this is valid as we have effectively introduced two new variables and equations by multiplying Eq.(3.27) by $a$ and $a^{\dagger}$. So we can reuse our original equations to help eliminate one of the variables. Later on it would be necessary to check that the states we calculate actually solve the original equations. From this we gain an equation for $\left|\psi_{E}^{+}\right\rangle$,

$$
\begin{equation*}
\left|\psi_{E}^{+}\right\rangle=\left\{\frac{-1}{\hbar^{2} \mathcal{E} g}\left[\left(E-i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right)^{2}-\frac{\hbar^{2} \Delta_{a}^{2}}{4}\right]+\frac{g}{\mathcal{E}}\right\}\left|\psi_{E}^{-}\right\rangle \tag{3.31}
\end{equation*}
$$

From this we can eliminate $\left|\psi_{E}^{+}\right\rangle$from the equations and gain a single equation in terms of $\left|\psi_{E}^{-}\right\rangle$

$$
\begin{align*}
& \left\{\left[\left(\frac{E}{g}\left(a^{\dagger}-a\right)+i \frac{E}{\hbar g}\right)^{2}+\frac{\Delta_{a}^{2}}{4 g^{2}}+a a^{\dagger}\right]\right. \\
& \left.\left[\left(\frac{E}{g}\left(a^{\dagger}-a\right)+i \frac{E}{\hbar g}\right)^{2}+\frac{\Delta_{a}^{2}}{4 g^{2}}+a^{\dagger} a\right]+\frac{\mathcal{E}^{2}}{g^{2}}\right\}\left|\psi_{E}^{-}\right\rangle=0 \tag{3.32}
\end{align*}
$$

Firstly, note that this equation is quartic in the creation and annihilation operators. Fortunately in this case we are able to factorise the above expression into two commuting quadratic operators. This will not be possible for the $\Delta_{c} \neq 0$ case. The factorised operators are

$$
\begin{align*}
\hat{O}_{p}(E) & =\left[\frac{\mathcal{E}}{g}\left(a^{\dagger}-a\right)+i \frac{E}{\hbar g}\right]^{2}+\frac{\Delta_{a}^{2}}{4 g^{2}}+\frac{a a^{\dagger}+a^{\dagger} a}{2}+\frac{1}{2} \sqrt{1-\frac{4 \mathcal{E}^{2}}{g^{2}}} \\
\hat{O}_{m}(E) & =\left[\frac{\mathcal{E}}{g}\left(a^{\dagger}-a\right)+i \frac{E}{\hbar g}\right]^{2}+\frac{\Delta_{a}^{2}}{4 g^{2}}+\frac{a a^{\dagger}+a^{\dagger} a}{2}-\frac{1}{2} \sqrt{1-\frac{4 \mathcal{E}^{2}}{g^{2}}} \tag{3.34}
\end{align*}
$$

So we seek solutions to

$$
\begin{equation*}
\hat{O}_{p}(E) \hat{O}_{m}(E)\left|\psi_{E}^{-}\right\rangle=0 \tag{3.35}
\end{equation*}
$$

From this we see that the solution will take the form

$$
\begin{equation*}
\left|\psi_{E}^{-}\right\rangle=c_{p}\left|\psi_{E, p}^{-}\right\rangle+c_{m}\left|\psi_{E, m}^{-}\right\rangle \tag{3.36}
\end{equation*}
$$

where $\left|\psi_{E, p}^{-}\right\rangle$and $c_{m}\left|\psi_{E, m}^{-}\right\rangle$are the solutions to

$$
\begin{align*}
\hat{O}_{p}\left|\psi_{E, p}^{-}\right\rangle & =0  \tag{3.37}\\
\hat{O}_{m}\left|\psi_{E, m}^{-}\right\rangle & =0 \tag{3.38}
\end{align*}
$$

In order to solve these equations we apply squeezing and displacement transformations to transform these equations into the eigenvalue equations of a harmonic oscillator. The explicit details of the transformation are contained in Appendix(A). We multiply Eqs. $(3.37),(3.38)$ by $S^{\dagger}(\eta) D^{\dagger}(\alpha)$ and insert $D(\alpha), S(\eta)$ and their adjoints as required. We transform the creation and annihilation operators according to

$$
\begin{align*}
S^{\dagger}(\eta) D^{\dagger}(\alpha) a D(\alpha) S(\eta) & =a \cosh \eta+a^{\dagger} \sinh \eta+\alpha  \tag{3.39}\\
S^{\dagger}(\eta) D^{\dagger}(\alpha) a^{\dagger} D(\alpha) S(\eta) & =a^{\dagger} \cosh \eta+a \sinh \eta+\alpha^{*} \tag{3.40}
\end{align*}
$$

with the parameters of the transformation given by

$$
\begin{align*}
\eta & =r \\
e^{2 r} & =\sqrt{1-4 \mathcal{E}^{2} / g^{2}}  \tag{3.41}\\
\alpha & =\frac{-2 i E \mathcal{E}}{\hbar g^{2}}\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right) \tag{3.42}
\end{align*}
$$

Then our transformed eigenvalue problems are given by

$$
\begin{gather*}
a^{\dagger} a\left|\tilde{\psi}_{E, p}^{-}\right\rangle=\left[\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right)^{-1}\left(\frac{E^{2}}{\hbar^{2} g^{2}}\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right)^{-2}+\frac{\Delta_{a}^{2}}{4 g^{2}}\right)+1\right]\left|\tilde{\psi}_{E, p}^{-}\right\rangle  \tag{3.43}\\
a^{\dagger} a\left|\tilde{\psi}_{E, m}^{-}\right\rangle=\left[\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right)^{-1}\left(\frac{E^{2}}{\hbar^{2} g^{2}}\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right)^{-2}+\frac{\Delta_{a}^{2}}{4 g^{2}}\right)\right]\left|\tilde{\psi}_{E, m}^{-}\right\rangle \tag{3.44}
\end{gather*}
$$

where

$$
\begin{equation*}
\left|\tilde{\psi}_{E}\right\rangle=S^{\dagger}(r) D^{\dagger}(\alpha)\left|\psi_{E}\right\rangle \tag{3.45}
\end{equation*}
$$

From Eq.(3.43) we see that the eigenstates $\left|\tilde{\psi}_{E}\right\rangle$ are the Fock states. Hence the energy eigenvalues are given by

$$
\begin{equation*}
E_{n}= \pm \hbar g\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right)^{1 / 2} \sqrt{\frac{\Delta_{a}^{2}}{4 g^{2}}+n\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right)} \tag{3.46}
\end{equation*}
$$

The other equation yields the same spectrum of energies, with $n \rightarrow n-1$. The eigenstates of the two equations are given by

$$
\begin{align*}
\left|\tilde{\psi}_{n, p}^{-}\right\rangle & =|n-1\rangle \quad n=1,2, \ldots  \tag{3.47}\\
\left|\tilde{\psi}_{n, m}^{-}\right\rangle & =|n\rangle \quad n=1,2, \ldots  \tag{3.48}\\
\left|\tilde{\psi}_{0}\right\rangle & =|0\rangle \tag{3.49}
\end{align*}
$$

Following on from the expressions utilised in the section on the laser coupling to the atom, it is apparent that the expressions for the eigenstates are likely to be exceedingly messy with little return for a large investment of time. I will outline how the eigenstates are formed as I have elected not to pursue this as it would be an exercise in tedious algebra for little gain. To find the eigenstates in the Schrödinger picture it is necessary to invert the displacement and squeezing transformations on the states. From here Eq.(3.31) is used to express $\left|\psi_{E}^{+}\right\rangle$in terms of these states. The constants $c_{p}$ and $c_{m}$ are fixed via the normalisation requirement.

In an analogous fashion to the case where the laser is driving the atom, these quasi-energies define shifts of energy levels around the energy eigenvalues of the transforming Hamiltonian. This ladder of quasi-energies is superimposed on every energy level of the bare Hamiltonian. In Figure (3.1) the spectrum is shown for the lowest lying energy eigenvalues up to a driving amplitude of $\mathcal{E}=g / 2$. The reason being that for $\mathcal{E}=g / 2$ all of the discrete energies have converged to zero. At the critical point we have infinitely many energy levels all converging onto one point. This implies that above this this critical point we have an infinite number of states at each point, i.e have a continuous spectrum.


Figure 3.1: Quasi-energies of resonant system, laser driving the cavity

This can be related to the spectra of the operators in Eq.(3.1). The terms in $a^{\dagger} a$ and $a^{\dagger} \sigma_{-}-a \sigma_{+}$have a discrete spectrum as shown in Section 2.1. The driving term $a^{\dagger}-a$ has a continuous spectrum as may be verified by noting that it is proportional to $\hat{X}_{2}$, one of the quadrature operators which possesses a constant spectrum. The threshold condition can be viewed as varying between the discrete spectrum of the Jaynes Cummings Hamiltonian and the continuous spectrum of the driving term.

### 3.3 Cavity Detuning, Cavity Driving

In this section I will show how the methods applied above break down when applied to the case of an off resonant cavity i.e. $\Delta_{c} \neq 0$. The basic problem can be seen by considering $E \rightarrow E-\hbar \Delta_{a} a^{\dagger} a$ in the original expressions for $\hat{O}_{p, m}(E)$ in $\mathrm{Eq}(3.33)$. This leads to terms quartic in the creation and annihilation operators. From there, no further progress is possible. For simplicity we will set $\Delta_{a}=0$. Restating Eq.(3.1)
$H=\hbar \omega_{c} a^{\dagger} a+\frac{\hbar \omega_{L}}{2} \sigma_{z}+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right)+i \hbar \mathcal{E}\left[\binom{\sigma_{+}}{a^{\dagger}} e^{-i \omega_{L} t}-\binom{\sigma_{-}}{a} e^{i \omega_{L} t}\right]$
For the case of cavity driving we have the coupled equations for the states

$$
\begin{align*}
& {\left[\hbar \Delta_{c} a^{\dagger} a-E+i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right]\left|\psi_{E}^{+}\right\rangle-i \hbar g a\left|\psi_{E}^{-}\right\rangle=0}  \tag{3.51}\\
& {\left[\hbar \Delta_{c} a^{\dagger} a-E+i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right]\left|\psi_{E}^{-}\right\rangle+i \hbar g a^{\dagger}\left|\psi_{E}^{+}\right\rangle=0} \tag{3.52}
\end{align*}
$$

As before we multiply by $a^{\dagger}$ and $a$ respectively to arrive at equations

$$
\begin{equation*}
\left(\left[\hbar \Delta_{c}\left(a^{\dagger} a-1\right)-E+i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right] a^{\dagger}+i \hbar \mathcal{E}\right)\left|\psi_{E}^{+}\right\rangle-i \hbar g a^{\dagger} a\left|\psi_{E}^{-}\right\rangle=0 \tag{3.53}
\end{equation*}
$$

$$
\begin{equation*}
\left(\left[\hbar \Delta_{c}\left(a^{\dagger} a+1\right)-E+i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right] a+i \hbar \mathcal{E}\right)\left|\psi_{E}^{-}\right\rangle+i \hbar g a a^{\dagger}\left|\psi_{E}^{+}\right\rangle=0 \tag{3.54}
\end{equation*}
$$

Using expressions for $a\left|\psi_{E}^{-}\right\rangle$and $a^{\dagger}\left|\psi_{E}^{+}\right\rangle$from Eq.(3.51) we can gain an expression for $\left|\psi_{E}^{+}\right\rangle$:

$$
\begin{align*}
\left|\psi_{E}^{+}\right\rangle= & \frac{i}{\hbar g}\left[\hbar \Delta_{c}\left(a^{\dagger} a-1\right)-E+i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right] \\
& {\left[\hbar \Delta_{c} a^{\dagger} a-E+i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right]\left|\psi_{E}^{-}\right\rangle-i \hbar g a^{\dagger} a\left|\psi_{E}^{-}\right\rangle } \tag{3.55}
\end{align*}
$$

This yields the rather awe inspiring eigenvalue problem for $\left|\psi_{E}^{-}\right\rangle$. Rather than directly express that rather messy problem I will show one of the analogous operators to $\hat{O}(E)$. In this case we have something like

$$
\begin{equation*}
\hat{O}(E)=\left[\hbar \Delta_{c}\left(a^{\dagger} a \pm 1\right)-E+i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right]\left[\hbar \Delta_{c} a^{\dagger} a-E+i \hbar \mathcal{E}\left(a^{\dagger}-a\right)\right] \tag{3.56}
\end{equation*}
$$

As promised the expression is quartic in the creation and annihilation operators. No one has yet developed any techniques for factorising arbitrary quartic terms in $a, a^{\dagger}$ into quadratic ones. We are also unable to directly solve quartic expressions like these, hence no further progress is possible.

A similar problem holds for the case of coupling to the atom. This may be brought out by considering a displacement transformation. We start from

$$
\begin{equation*}
H=\hbar \Delta_{a} a^{\dagger} a+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right)+i \hbar \mathcal{E}\left(\sigma_{+}-\sigma_{-}\right) \tag{3.57}
\end{equation*}
$$

Now applying a displacement

$$
D^{\dagger}(\alpha) a D(\alpha)=a+\alpha
$$

our Hamiltonian becomes

$$
\begin{align*}
H^{\prime}= & \hbar \Delta_{c}\left(a^{\dagger}+\alpha^{*}\right)(a+\alpha)+i \hbar g\left[\left(a^{\dagger}+\alpha^{*}\right) \sigma_{-}-(a+\alpha) \sigma_{+}\right]+i \hbar \mathcal{E}\left(\sigma_{+}-\sigma_{-}\right)+|\alpha|^{2} \\
= & \hbar \Delta_{c}\left(a^{\dagger} a+|\alpha|^{2}\right)+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right) \\
& +\hbar \Delta_{c}\left(\alpha a^{\dagger}+\alpha^{*} a\right)+(i \hbar \mathcal{E}-i \hbar g \alpha)\left(\sigma_{+}-\sigma_{-}\right) \tag{3.58}
\end{align*}
$$

So if we choose $\alpha=\mathcal{E} / g$ then we have a similar form of operators as exists in the case of coupling to the cavity and similar problems will hold here.

$$
\begin{equation*}
H=\hbar \Delta_{c}\left(a^{\dagger} a+\mathcal{E}^{2} / g^{2}\right)+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right)-\frac{\hbar \mathcal{E} g}{\Delta_{c}}\left(a^{\dagger}+a\right) \tag{3.59}
\end{equation*}
$$

Consequently no further exact analytical work is possible at this stage.

## Chapter 4

## Perturbation Theory

This chapter outlines a few attempts to derive approximate analytical results to aid understanding of the numerical simulations. Typically these have treated the $\hbar \Delta_{c} a^{\dagger} a$ term as a perturbation upon the exact eigenstates found in the previous chapter. These shifts are found in the case where $\Delta_{a}=0$. The energy shifts are found to first order in $\Delta_{c}$.

### 4.1 Cavity Detuning, Atomic Driving

Our initial eigenstates in the limit that $\Delta_{c}=0$ are

$$
\begin{align*}
\left|\psi_{n, u}\right\rangle & =\frac{1}{\sqrt{2}}(|\mathcal{E} / g ; n-1\rangle|+\rangle+i|\mathcal{E} / g ; n\rangle|-\rangle)  \tag{4.1}\\
\left|\psi_{n, l}\right\rangle & =\frac{1}{\sqrt{2}}(|\mathcal{E} / g ; n-1\rangle|+\rangle-i|\mathcal{E} / g ; n\rangle|-\rangle) \tag{4.2}
\end{align*}
$$

where $|\alpha ; n\rangle=D(\alpha)|n\rangle$. Seeing as these states are non-degenerate we may proceed with simple first order perturbation theory. In general the first order energy shift for a state of the unperturbed Hamiltonian, $|n\rangle$, is given by

$$
E_{n}^{(1)}=\lambda\langle n| V|n\rangle
$$

where $\lambda$ is a small parameter and V is the perturbation. In our case we have

$$
\begin{equation*}
E_{n, \xi}^{(1)}=\Delta_{c}\left\langle\psi_{n, \xi}\right| a^{\dagger} a\left|\psi_{n, \xi}\right\rangle \tag{4.3}
\end{equation*}
$$

We now insert the form of the exact eigenstates, and use the Displacement transformation on $a^{\dagger} a$ to arrive at our result. Beginning with the ground state we find

$$
\begin{align*}
E_{0}^{(1)} & =\langle 0| D^{\dagger}(\alpha) a^{\dagger} a D(\alpha)|0\rangle \\
& =\langle 0| a^{\dagger} a+\alpha^{\star} a+\alpha a^{\dagger}+|\alpha|^{2}|0\rangle \\
& =|\alpha|^{2}=\mathcal{E}^{2} / g^{2} \tag{4.4}
\end{align*}
$$

Similar for the excited states

$$
E_{n, u}^{(1)}=\frac{1}{2}(\langle n-1,+|-i\langle n,-|) D^{\dagger}(\alpha) a^{\dagger} a D(\alpha)(|n-1,+\rangle+i|n,-\rangle)
$$

$$
\begin{align*}
& =\frac{1}{2}(\langle n-1,+|-i\langle n,-|)\left(a^{\dagger}+\alpha^{*}\right)(a+\alpha)(|n-1,+\rangle+i|n,-\rangle) \\
& =(\langle n-1,+|-i\langle n,-|)\left(\frac{a^{\dagger} a+\alpha^{*} a+\alpha a^{\dagger}|\alpha|^{2}}{2}\right)(|n-1,+\rangle+i|n,-\rangle) \\
& =n-1 / 2+|\alpha|^{2}=n-1 / 2+\mathcal{E}^{2} / g^{2} \tag{4.5}
\end{align*}
$$

An identical result also holds for $E_{n, l}^{(1)}$. This simple result will be used in the next section to provide comparison with the numerical simulations.

### 4.2 Cavity Detuning, Cavity Driving

In this case the results are of limited usefulness. Due to the transition between discrete and continuous energy spectra, the exact solutions are only valid for $0 \leq$ $\mathcal{E}<g / 2$. That said, due to the overall difficulty of interpreting the numerical results, any light shed is useful.

In a similar manner to the previous section we use the fully resonant $\left(\omega_{a}=\right.$ $\omega_{L}=\omega_{c}$ ) eigenstates as found in Section IV of [2]. We have as our basis states

$$
\begin{align*}
\left|\chi_{0}\right\rangle & =|r, 0 ; 0\rangle|M\rangle  \tag{4.6}\\
\left|\chi_{n, u}\right\rangle & =\frac{1}{\sqrt{2}}\left(\left|r, \beta\left(E_{n, u}^{(0)}\right) ; n\right\rangle|P\rangle+i\left|r, \beta\left(E_{n, u}^{(0)}\right) ; n\right\rangle|M\rangle\right)  \tag{4.7}\\
\left|\chi_{n, l}\right\rangle & =\frac{1}{\sqrt{2}}\left(\left|r, \beta\left(E_{n, l}^{(0)}\right) ; n\right\rangle|P\rangle-i\left|r, \beta\left(E_{n, l}^{(0)}\right) ; n\right\rangle|M\rangle\right) \tag{4.8}
\end{align*}
$$

where

$$
\begin{align*}
|\eta, \alpha ; n\rangle & =D(\alpha) S(\eta)|n\rangle \\
|P\rangle & =\frac{1}{\sqrt{2}}\left[\left(1+\sqrt{1-\frac{4 \mathcal{E}^{2}}{g^{2}}}\right)^{\frac{1}{2}}|+\rangle-\left(1-\sqrt{1-\frac{4 \mathcal{E}^{2}}{g^{2}}}\right)^{\frac{1}{2}}|-\rangle\right]  \tag{4.9}\\
|M\rangle & =\frac{1}{\sqrt{2}}\left[\left(1+\sqrt{1-\frac{4 \mathcal{E}^{2}}{g^{2}}}\right)^{\frac{1}{2}}|-\rangle-\left(1-\sqrt{1-\frac{4 \mathcal{E}^{2}}{g^{2}}}\right)^{\frac{1}{2}}|+\rangle\right] \tag{4.10}
\end{align*}
$$

where the parameters of squeezing and displacement transformations are

$$
\begin{equation*}
e^{2 r}=\sqrt{1-4 \mathcal{E}^{2} / g^{2}} \quad \beta(E)=\frac{-i E}{\hbar g} \frac{2 \mathcal{E} / g}{1-4 \mathcal{E}^{2} / g^{2}} \tag{4.11}
\end{equation*}
$$

and the unperturbed energies are defined by

$$
\begin{equation*}
E_{n, \xi}^{(0)}= \pm \hbar g \sqrt{n}\left[1-4 \mathcal{E}^{2} / g^{2}\right]^{3 / 4} \tag{4.12}
\end{equation*}
$$

Calculating the first order shifts leads to similar manipulations as used above, where we transform the operator $a^{\dagger} a$. The required transformations may be found in Appendix(A).

$$
\begin{align*}
E_{n, u}^{(1)}= & \frac{1}{2}(\langle n-1|\langle P|-i\langle n|\langle M|) \\
& \times\left(S^{\dagger}(\eta) D^{\dagger}(\alpha) a^{\dagger} a D(\alpha) S(\eta)\right)(|n-1\rangle|P\rangle+i|n\rangle|M\rangle) \tag{4.13}
\end{align*}
$$

From the appendix we have that

$$
\begin{align*}
\left(a^{\dagger} a\right)^{\prime} & =S^{\dagger}(\eta) D^{\dagger}(\alpha) a^{\dagger} a D(\alpha) S(\eta)=a^{\dagger} a \cosh 2 r+\frac{\cosh 2 r-1}{2} \\
& -\frac{\sinh 2 r}{2}\left(a^{2}+{a^{\dagger}}^{2}\right)+\alpha e^{r}\left(a^{\dagger}-a\right)+|\alpha|^{2} \tag{4.14}
\end{align*}
$$

which holds in the case where $\alpha^{*}=-\alpha$, as is the case here. From here we find that for two number states

$$
\begin{align*}
\langle n| S^{\dagger}(\eta) D^{\dagger}(\alpha) a^{\dagger} a D(\alpha) S(\eta)|m\rangle & =\left(n \cosh 2 r+|\alpha|^{2}+\frac{\cosh 2 r-1}{2}\right) \delta_{n, m} \\
& -\frac{\sinh 2 r}{2}\left(\sqrt{m(m-1)} \delta_{n, m-2}+\sqrt{(m+1)(m+2)} \delta_{n, m+2}\right) \\
& +\alpha e^{r}\left(\sqrt{m+1} \delta_{n, m+1}-\sqrt{m} \delta_{n, m-1}\right) \tag{4.15}
\end{align*}
$$

For the ground state we have

$$
\begin{align*}
E_{0}^{(1)} & =\left\langle\chi_{0}\right| a^{\dagger} a\left|\chi_{0}\right\rangle \\
& =\langle 0| S^{\dagger}(\eta) a^{\dagger} a S(\eta)|0\rangle \\
& =\frac{\cosh 2 r-1}{2} \tag{4.16}
\end{align*}
$$

where we have used $D(0)=1$ and Eq.(4.15). The shifts for the excited states are given by:

$$
\begin{align*}
E_{n, u}^{(1)}= & 1 / 2(\langle n-1|\langle P|-i\langle n|\langle M|)\left(S^{\dagger}(\eta) D^{\dagger}(\alpha) a^{\dagger} a D(\alpha) S(\eta)\right)(|n-1\rangle|P\rangle+i|n\rangle|M\rangle) \\
= & 1 / 2\left[\langle n-1|\left(a^{\dagger} a\right)^{\prime}|n-1\rangle-i\langle n|\left(a^{\dagger} a\right)^{\prime}|n-1\rangle\langle M \mid P\rangle\right. \\
& \left.+i\langle n-1|\left(a^{\dagger} a\right)^{\prime}|n\rangle\langle P \mid M\rangle+\langle n|\left(a^{\dagger} a\right)^{\prime}|n\rangle\right] \\
= & (n-1 / 2) \cosh 2 r+|\alpha|^{2}+\frac{\cosh 2 r-1}{2}+\frac{2 i \mathcal{E} \alpha}{g} e^{r} \sqrt{n} \tag{4.17}
\end{align*}
$$

Applying similar steps we also find that

$$
\begin{equation*}
E_{n, l}^{(1)}=(n-1 / 2) \cosh 2 r+|\alpha|^{2}+\frac{\cosh 2 r-1}{2}-\frac{2 i \mathcal{E}}{g} \alpha e^{-r} \sqrt{n} \tag{4.18}
\end{equation*}
$$

If we substitute in the parameters for $\beta(E)$ and $e^{2 r}$ then we arrive at the following form for the first order energy shifts

$$
\begin{gather*}
E_{n, u}^{(1)}=\frac{n}{2}\left(1+\frac{8 \mathcal{E}^{2}}{g^{2}}\right)\left[\left(1-\frac{4 \mathcal{E}}{g^{2}}\right)^{\frac{1}{2}}+\left(1-\frac{4 \mathcal{E}}{g^{2}}\right)^{-\frac{1}{2}}\right]-\frac{1}{2}  \tag{4.19}\\
E_{n, l}^{(1)}=\frac{n}{2}\left[\left(1+\frac{8 \mathcal{E}^{2}}{g^{2}}\right)^{\frac{1}{2}}\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right)^{\frac{1}{2}}+\left(1-\frac{8 \mathcal{E}^{2}}{g^{2}}\right)^{\frac{1}{2}}\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right)^{-\frac{1}{2}}\right]-\frac{1}{2} \tag{4.20}
\end{gather*}
$$

It should be immediately pointed out that these results depend upon $e^{-2 r}=$ $\left(1-4 \mathcal{E}^{2} / g^{2}\right)^{-1 / 2}$ and so diverge at $\mathcal{E}=g / 2$. Another related weakness is that at $\mathcal{E}=g / 2$ all of the zeroth order eigenvalues become degenerate and nondegenerate perturbation theory breaks down. So these results are only valid for $\mathcal{E}<g / 2$. As a result of these weaknesses I have been loathe to extend these results to second order as I doubt the results would be useful. Despite these shortcomings, they do lend some insight into the numerical simulations.

## Chapter 5

## Numerical Simulations

This chapter covers numerical simulations I have undertaken. I have compared numerical simulations and the exact analytical solutions as a check on the simulations. The simulations are then extended to cases where there are no exact analytical results and the only comparisons are from perturbation theory. I have focused on the regime where $\Delta_{c}$ is small so that I may compare the numerical simulations with the results from perturbation theory. The numerical simulations were undertaken in Matlab.

The basic procedure was to select an appropriate set of basis states and then express the operators in the Hamiltonian as matrices in this basis. The resulting matrix representing the Hamiltonian is diagonalised numerically for a given set of parameters $g, \Delta_{a}, \Delta_{c}$ and $\mathcal{E}$. This was done using Matlab's built in solving routine to extract the energy eigenvalues and eigenvectors.

At some point I must mention units. I have elected to set $\hbar=1$ explicitly and have usually left $g=1$ as we can scale the relative size of terms by varying the other two parameters, $\mathcal{E}$ and $\Delta_{a}$ or $\Delta_{c}$.

I initially planned on using two sets of basis states. The first set of states were the separate eigenstates of the terms $a^{\dagger} a$ and $\sigma_{z}$, combinations of number and atomic eigenstates, $|n, \pm\rangle$. These states have the advantage of being the simplest to implement.

The other set of states were the dressed states. These diagonalise the Jaynes Cummings Hamiltonian

$$
\begin{equation*}
\hbar \Delta_{c} a^{\dagger} a+\hbar \Delta_{a} / 2 \sigma_{z}+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right) \tag{5.1}
\end{equation*}
$$

These states obviously diagonalise the full Hamiltonian for $\mathcal{E}=0$. Unfortunately my implementation of these introduced spurious eigenvalues that I could not rectify. In addition these calculations were not substantially faster nor the results obviously superior. Hence all simulations were undertaken with a number state basis.

### 5.1 Atomic Detuning, Cavity Driving

In this section I will compare the case where we have full analytical solutions with their numerical counterparts. This lends the numerical simulations a semblance of validity for the later purely numerical work. In the case where $\Delta_{c}=0$,


Figure 5.1: Energy Spectrum for Atomic Detuning, Cavity Coupling. $\Delta_{a}=0.1$, basis size $=300$, Dotted Lines analytical solution, Crosses Numerical solution
$\Delta_{a} \neq 0$ we have from Eq. (3.46),

$$
\begin{align*}
H & =\frac{\hbar \Delta_{a}}{2} \sigma_{z}+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right)+i \hbar \mathcal{E}\left(a^{\dagger}-a\right)  \tag{5.2}\\
E_{n} & = \pm \hbar g\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right)^{1 / 2} \sqrt{\frac{\Delta_{a}^{2}}{4 g^{2}}+n\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right)^{1 / 2}} . \tag{5.3}
\end{align*}
$$

As noted earlier these energies are discrete for $\mathcal{E}<g / 2$, but for $\mathcal{E} \geq g / 2$ they are continuous. From Figures (5.1) and (5.2) we see that the analytical results agree closely with their numerical counterparts below the critical point. Above this point the predicted energies, given by Eq. (5.3), become complex. This breakdown is consistent with the earlier suggestion that in this regime the spectrum is continuous. If we look at the numerical solutions these remain close to zero above the critical point. This is in line with the idea of the energies forming a continuum. Since we have a finite dimensional approximation they remain discrete. As the basis size increases, these values get closer to zero. It has been suggested that the spread between energy levels gives an idea as to the behaviour of the density of states. Reassured as to the validity of these numerical simulations we progress on to the case where $\Delta_{c} \neq 0$.

### 5.2 Cavity Detuning, Cavity Driving

We now progress to considering an off resonant cavity, $\Delta_{c} \neq 0$. In this regime our only tools are numerical simulations and perturbation theory as there are no exact results. We begin by covering some qualitative aspects of the spectrum before moving on to the simulations. The Hamiltonian is given by

$$
H=\hbar \Delta_{c} a^{\dagger} a+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right)-i \hbar \mathcal{E}\left(a^{\dagger}-a\right)
$$

This has a discrete spectrum which may be considered by enacting a displacement transformation, $D(\alpha)$ with $\alpha=i \mathcal{E} / \Delta_{c}$, on the Hamiltonian. The mathematics is very similar to where I highlighted the similarity between cavity and


Figure 5.2: Energy Spectrum for Atomic Detuning, Cavity Coupling. $\Delta_{a}=1$, basis size $=300$, Dotted Lines analytical solution, Crosses Numerical solution
atomic coupling in Section (3.3). We find the following form for the displaced Hamiltonian

$$
\begin{equation*}
H=\hbar \Delta_{c} a^{\dagger} a+i \hbar g\left(a^{\dagger} \sigma_{-}-a \sigma_{+}\right)-\frac{\hbar \mathcal{E}^{2}}{\Delta_{c}}-\frac{\hbar \mathcal{E} g}{\Delta_{c}}\left(\sigma_{+}+\sigma_{-}\right) \tag{5.4}
\end{equation*}
$$

The final term, $\sigma_{+}+\sigma_{-}$, has eigenstates $|+\rangle+|-\rangle$and $|+\rangle-|-\rangle$. So all the terms in our Hamiltonian have a discrete eigenvalue spectrum. Thus we expect the eigenvalue spectrum of the whole Hamiltonian to be discrete. In plots where $\Delta_{c} \ll 1$ the discrete character of the spectrum is not obvious for all $\mathcal{E}$ and must be taken on faith.

One of the striking features of the plots of the spectra is that there appear to be two sets of spectra superimposed on one another. The first set is analogous to the low lying eigenvalues of the resonant case. The second set is closely spaced and these eigenvalues cross over the first set. As a result the first set of eigenvalues must be inferred from the plots. These features can be understood if we examine the eigenvalues of the Jaynes Cummings model. For the case $\mathcal{E}=0$ our Hamiltonian reduces to the Jaynes-Cummings model with eigenvalues given by

$$
\begin{equation*}
E_{n, \pm}=\hbar \Delta_{c}\left[(n+1 / 2) \pm \sqrt{\frac{1}{4}+\frac{g^{2}(n+1)}{\Delta_{c}^{2}}}\right] \tag{5.5}
\end{equation*}
$$

If we look at $E_{n,-}$ we see that the energy depends on $n$ and $\sqrt{n}$. For small $n, n \ll$ $\sqrt{n}$ so the negative term will dominate and cause the energies to decrease. As $n$ increases the positive term will then dominate and the energies will increase. We can solve for the value of $n$ when $E_{n,-}$ ascends through zero. We find that this occurs for $n=g^{2} / \Delta_{c}^{2}$.

From Eq. (5.5) we can also calculate the spacing between states and invert to get an approximate density of states

$$
\begin{equation*}
\frac{d n}{d E_{ \pm}}=\frac{\sqrt{n}}{\hbar \Delta_{c} \sqrt{n} \pm \hbar g} \tag{5.6}
\end{equation*}
$$



Figure 5.3: Approximate Density of States for Jaynes-Cummings Model. For $g=1, \Delta_{c}=0.01$.

A typical example is included in Figure (5.3). In general as the size of our basis increases, so does the agreement for the lower lying eigenvalues. As the density of states is large for states given by $E_{n,-}$, and the energies decrease to a minimum then increase with n , we see a dense band of states for $E<$ 0 . As $\mathcal{E}$ increases these eigenvalues spread out and cross over the eigenvalues corresponding to smaller $n$.

The best example of this phenomenon is in Figure (5.7) where the basis size is large enough that the eigenvalues corresponding to $E_{n,-}$ have become positive and have swamped the entire plot. This doesn't occur for the other plots as in those cases the basis size is not large enough to include states such that $E_{n,-}$ becomes positive.

An unfortunate side effect of the crossing of energy eigenvalues is that it is difficult to track a particular eigenvalue. As we vary $\mathcal{E}$ as the indices corresponding to a given eigenvalue vary and hence plotting the eigenvalue corresponding to one state is problematic. Hence I have had to resort to plotting the entire set of eigenvalues to track those that are most similar to their resonant counterparts.

A possible method of tracking the eigenvalues is to examine the composition of the eigenstates. If we seek the state which bears the most similarity to our chosen state as we vary from one parameter step to the next, then preliminary checks suggest we could track the individual eigenvalues. So far the attempts to implement this have been unreliable and hence have been omitted.

We now progress to discussing in detail some of the numerical simulations.
In Figure (5.4), we see that for $\mathcal{E}>g / 2$ around $\mathrm{E}=0$ there is a region of similarity to the results for the resonant case. Much like that case there is no crossing of levels and the levels spread apart in an apparently discrete fashion, although these energies are shifted upwards. In Figure (5.5) we see that even for $\Delta_{c}=0.01$ the perturbation energies have a rather poor agreement with their numerical counterparts. Which suggests that either I have made an error somewhere or that the perturbation expansion is not accurate.

Looking at Figures (5.4), (5.6) and (5.7) we see that the positive eigenvalues for low n all pass through $E=0$ for $\mathcal{E}>g / 2$. As $\Delta_{c} \rightarrow 0$ the crossing points of these eigenvalues get closer to together and closer to $\mathcal{E}=g / 2$ while magnitude


Figure 5.4: Numerical solutions for $\Delta_{c}=0.01$. Number state basis size $=300$. Note traces of energy eigenvalues buried within energies crossing over


Figure 5.5: Numerical solutions for $\Delta_{c}=0.01$. Number state basis size $=300$. Numerical solutions solid line, perturbation solution dashed line.


Figure 5.6: $\Delta_{c}=0.05$ Numerical solutions. Number state basis size $=300$


Figure 5.7: $\Delta_{c}=0.075$ Numerical solutions. Number state basis size $=400$. Note dense band of states extends beyond $\mathrm{E}=0$


Figure 5.8: Plot of energies in perturbation theory. $\Delta_{c}=0.05$. Note region where there are no energy eigenvalues.
of their slope increases. I suspect that the continuum arises as the zero point of the eigenvalues meet and their slope goes to $-\infty$. Conversely as $\Delta_{c}$ grows, they move apart from one another eventually going over to some form of ladder. The behaviour around $\mathrm{E}=0$ for Figure (5.4) is quite close to that of the case on resonance which either suggests something similar to a continuum in that region or a more complicated dependence on $\Delta_{c}$ than I am proposing.

One other feature of interest common to Figures (5.4), (5.6) and (5.7) is how the negative eigenvalues converge onto one another and then peter out. In all figures we see what appears to be a hyperbola or a parabola defining a region where there are no negative energy eigenvalues corresponding to low $n$. As $\Delta_{c}$ increases so does the width of the curve. In all cases the apex is located at $\mathcal{E}=1, E=0$. While the feature recurs elsewhere, it may be a fault in the numerics. A mark of doubt is that the parameters of the curve are dependent on the size of our basis. If we increase the size of the basis the parameters of this curve shift slightly, implying that this may be an effect of using a truncated basis. That this also occurs for the case where $\Delta_{c}=0$, reinforces that idea.

As useless as the perturbation theory results may appear to be they do recreate a similar phenomenon. The perturbation theory results generate an envelope below which there are no eigenvalues. As $\Delta_{c}$ varies eigenvalues corresponding to larger n increase in energy more quickly. These eigenvalues increase, leaving lower and lower $n$ values to define the envelope until eventually $\mathrm{E}=0$. Once the eigenvalues leave the envelope they do not cross each other again. An example is provided in Figure (5.8). This suggests the possibility that these negative energy eigenvalues have "diverged" in a manner similar to the perturbation theory results and hence we have lost the distinctive track we have been following as there no more crossings.


Figure 5.9: Energy vs. Cavity Detuning. $2 \mathcal{E} / g=0.5$ Number state basis size $=$ 300


Figure 5.10: Energy vs. Cavity Detuning. $2 \mathcal{E} / g=1$ Number state basis size $=$ 300


Figure 5.11: Energies from perturbation theory. $2 \mathcal{E} / g=0.5$

In Figures (5.9) and (5.10) we are varying $\Delta_{c}$ for fixed $\mathcal{E}$. The salient features are to note are that as in the plot of energy vs. $\mathcal{E}$, these plots also show what look to be tracks of energy eigenvalues with another spectrum overlaid on top. The lower energy eigenvalues obey a linear dependence in $\Delta_{c}$, as may be expected from perturbation theory. Note however that the other eigenvalues seem to obey a much more complicated dependence on $\Delta_{c}$.

In Figure (5.10) we have evaluated this plot at the critical driving strength $\mathcal{E}=g / 2$. So as $\Delta_{c} \rightarrow 0$ the discrete energies converge onto one another. At this point the perturbation theory treatment is invalid. Unlike the previous plot the decay to zero is not linear in $\Delta_{c}$. The decay to zero suggests a power law dependence somewhat analogous to that found for the energies in Eq. (3.46). Not all energies converge onto $\mathrm{E}=0$ for $\Delta_{c}=0$ as they should which is an effect of having a finite basis size. Finally there appear to be no discrete energies for $E<0$. This appears to vindicate what was found in looking at the plots of energy vs. $\mathcal{E}$.

An interesting phenomenon common to both plots is the disappearance of negative eigenvalues as we increase $\Delta_{c}$. I suspect that this is due to the eigenvalues not crossing and hence we lose the tell-tale markers we have been using. If we look at the results from perturbation theory as plotted in Figure (5.11) we see that as $\Delta_{c}$ increases the negative energy eigenvalues stop crossing over one another. As this has been the means by which we have observed these, we lose the the tracks. For the case where $\mathcal{E}=g / 2$ this explanation is invalid and consequently I have little idea of what is happening here.

### 5.3 Cavity Detuning, Atomic Driving

Here we examine the case where the laser is directly driving the atoms which are inside an off-resonant cavity i.e. $\omega_{c} \neq \omega_{a}=\omega_{L}$. In this case the spectrum on resonance is independent of $\mathcal{E}$, unlike the case where the cavity is being driven. The introduction of cavity detuning does not introduce any remarkable new features. The spectrum remains discrete as is expected.

In Figure (5.12) we see that the perturbation results agree for $\Delta_{c}$ small. in


Figure 5.12: Energy eigenvalues vs. driving strength $(2 \mathcal{E} / g) . \Delta_{c}=0.01$. Number state basis size $=300$. Solid lines numerical simulation. Crosses perturbations theory results.


Figure 5.13: Energy eigenvalues vs. driving strength $(2 \mathcal{E} / g) . \Delta_{c}=0.1$. Number state basis size $=300$.


Figure 5.14: Energy vs Cavity Detuning. $2 \mathcal{E} / g=1$. Number state basis size $=$ 300. Dashed lines perturbation theory, dots numerical simulation.

Figure (5.13) we can see the basis size and $\Delta_{c}$ are large enough so that the $E_{n,-}$ have become positive and filled up the plot. The buried eigenvalues appear to have retained their quadratic dependence on $\mathcal{E}$.

In Figure (5.14) there is an example of a plot of Energy vs $\Delta_{c}$. In a similar manner to the previous case of cavity coupling the plots also become swamped with traces of lower eigenvalues. The perturbation theory results, while similar to the numerical spectrum, are probably invalid. Perturbation theory is valid only for small energy shifts. In this case where $E_{n, \xi} \sim n \Delta_{c}$ for large enough n , the shifts will be large and the results invalid.

## Chapter 6

## Conclusions

We have recalculated the energies of the driven Jaynes Cummings system including an atomic detuning. In the case of cavity detuning first order perturbation theory was used to find the first corrections to the energies. In the case of coupling to the cavity these were of limited usefulness as the analytical solutions are only valid for $0<2 \mathcal{E} / g<1$. Despite their limited validity they have been useful in providing qualitative arguments for what may be occuring.

As $\Delta_{c}$ and $\mathcal{E}$ increase the energies for different states appear to begin to cross over. This can explained by the Jaynes-Cummings model energies and the results from perturbation theory. This leads to rather messy plots with energy eigenvalues of low lying states needing to be inferred. Examining only these eigenvalues confirms the discrete nature of the spectrum. We found that in a region around $\mathcal{E}=g / 2$ there appear to be no negative quasienergies. This might be explained by similar behaviour exhibited by the perturbative results. A similar explanation is possible when we are examining the dependence on $\Delta_{c}$.

The one overriding feature of the discussion in Chapter 5 has been that is it overwhelming qualititative and tentative in its conclusions. This is is largely a result of being unable to track particular energy eigenvalues and hence having to deal with the entire spectrum. It follows that the top priority in completing this work is to devise a means by which we can track a particular eigenvalue. A second major task is to overhaul the code to improve efficiency as it is impractical to use basis sizes larger than 400 states.

At the end of all this it is achingly obvious that there is still a large area to explore in terms of numerical simulations. The above obstacles and poor time management have precluded carrying this work further. It is almost certain that the results presented here could be extended quite easily to resolve the questions left unanswered.

## Appendix A

## Squeezing and <br> Displacement Transformation

In this section I will set out the details of the transformation required in passing from the form of the operators in Eq. (3.33) to the harmonic oscillator. The transformation required is identical to that used in [2] as the extra term $\Delta_{a}^{2} / 4 g^{2}$ does nothing to change the transformation parameters. Nonetheless I will include details hidden in a few lines of work in the main body of the text. We start with the operators in the form

$$
\begin{equation*}
\hat{O}(E)=\left(\frac{\mathcal{E}}{g}\left(a^{\dagger}-a\right)+i \frac{E}{\hbar g}\right)^{2}+\frac{a a^{\dagger}+a^{\dagger} a}{2}+\frac{\Delta_{a}^{2}}{4 g^{2}} \tag{A.1}
\end{equation*}
$$

We neglect the constant terms involved in the full definition of $\hat{O}_{p}(E)$ and $\hat{O}_{m}(E)$ as our purpose is to transform the terms in $a$ and $a^{\dagger}$ into something similar to $a^{\dagger} a$. So the constant terms are irrelevant in determining the parameters of the transformation. Expanding this out we have
$\hat{O}=\frac{\mathcal{E}^{2}}{g^{2}}\left(a^{\dagger^{2}}+a^{2}\right)+\left(a^{\dagger} a+1 / 2\right)\left(1-\frac{2 \mathcal{E}^{2}}{g^{2}}\right)+\frac{2 i E \mathcal{E}}{\hbar g^{2}}\left(a^{\dagger}-a\right)-\frac{E^{2}-\hbar^{2} \Delta_{a}^{2} / 4}{\hbar^{2} g^{2}}$
We apply a combined squeezing and displacement transformation to the creation and annihilation operators.

$$
\begin{align*}
a^{\prime}=S^{\dagger}(r) D^{\dagger}(\alpha) a D(\alpha) S(r) & =a \cosh r+a^{\dagger} \sinh r+\alpha  \tag{A.3}\\
a^{\prime \dagger}=S^{\dagger}(r) D^{\dagger}(\alpha) a^{\dagger} D(\alpha) S(r) & =a^{\dagger} \cosh r+a^{\dagger} \sinh r+\alpha^{*} \tag{A.4}
\end{align*}
$$

Here we have used $\eta=r$ i.e. $\theta=0$. As a result we find that

$$
\begin{align*}
a^{\prime \dagger^{2}} & =\frac{1}{2} \cosh 2 r\left(a^{\dagger^{2}}+a^{2}\right)+\frac{1}{2}\left(a^{\dagger^{2}}-a^{2}\right) \\
& +\frac{2 a^{\dagger} a+1}{2} \sinh 2 r-2 \alpha\left(a^{\dagger} \cosh r+a \sinh r\right)+\alpha^{2} \tag{A.5}
\end{align*}
$$

$$
\begin{align*}
a^{\prime 2} & =\frac{1}{2} \cosh 2 r\left(a^{\dagger^{2}}+a^{2}\right)-\frac{1}{2}\left(a^{\dagger^{2}}-a^{2}\right) \\
& +\frac{2 a^{\dagger} a+1}{2} \sinh 2 r+2 \alpha\left(a \cosh r+a^{\dagger} \sinh r\right)+\alpha^{2}  \tag{A.6}\\
a^{\prime \dagger} a^{\prime} & =\frac{a^{\dagger} a+a a^{\dagger}}{2} \cosh 2 r-1 / 2+\frac{a^{\dagger^{2}}+a^{2}}{2} \sinh 2 r \\
& +\alpha\left(a^{\dagger}-a\right)(\cosh r-\sinh r)-\alpha^{2} \tag{A.7}
\end{align*}
$$

Now substituting into Eq. (A.2) and grouping together like terms we get

$$
\begin{align*}
\hat{O}^{\prime}(E)= & \left(a^{2}+a^{\dagger^{2}}\right)\left[\frac{\mathcal{E}^{2}}{g^{2}} \cosh 2 r+\left(\frac{1}{2}-\frac{\mathcal{E}^{2}}{g^{2}}\right) \sinh 2 r\right] \\
& +\left(2 a^{\dagger} a+1\right)\left[\frac{\mathcal{E}^{2}}{g^{2}} \sinh 2 r+\left(\frac{1}{2}-\frac{\mathcal{E}^{2}}{g^{2}}\right) \cosh 2 r\right] \\
& +\left(a^{\dagger}-a\right) e^{-r}\left[-2 \alpha \frac{\mathcal{E}^{2}}{g^{2}}+\alpha\left(1-\frac{2 \mathcal{E}^{2}}{g^{2}}\right)+\frac{2 i E \mathcal{E}}{\hbar g^{2}}\right] \\
& -\alpha^{2}\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right)-\frac{E^{2}-\hbar^{2} \Delta_{a}^{2} / 4}{\hbar^{2} g^{2}}-\frac{4 i E \mathcal{E}}{\hbar g^{2}} \alpha \tag{A.8}
\end{align*}
$$

From the above equation we are able to choose the parameters of the transformation so that the terms in $a^{2}+a^{\dagger^{2}}$ and $a^{\dagger}-a$ vanish. After a little manipulation we arrive at

$$
\begin{align*}
e^{2 r} & =\sqrt{1-\frac{4 \mathcal{E}^{2}}{g^{2}}}  \tag{A.9}\\
\alpha & =\frac{-2 i E \mathcal{E}}{\hbar g^{2}}\left(1-\frac{4 \mathcal{E}^{2}}{g^{2}}\right) \tag{A.10}
\end{align*}
$$

With this parameter choice the transformed operator $\hat{O}^{\prime}$ becomes

$$
\begin{align*}
\hat{O}^{\prime}(E) & =\left(a^{\dagger} a+1 / 2\right) e^{2 r}-\alpha\left(\frac{4 i E \mathcal{E}}{\hbar g^{2}}+\alpha e^{4 r}\right)-\frac{E^{2}-\hbar^{2} \Delta_{a}^{2} / 4}{\hbar^{2} g^{2}} \\
& =\left(a^{\dagger} a+1 / 2\right) e^{2 r}-\frac{E^{2}}{\hbar^{2} g^{2}} e^{-4 r}+\frac{\Delta_{a}^{2}}{4 g^{2}} \tag{A.11}
\end{align*}
$$

Now referring to our definition of $\hat{O}_{p}(E)$ and $\hat{O}_{m}(E)$ in Eq. (3.33), the transformed operators are given by

$$
\begin{gather*}
\hat{O}_{p, m}^{\prime}(E)=\hat{O}_{p, m} \pm \frac{1}{2} \sqrt{1-4 \mathcal{E}^{2} / g^{2}} \\
\hat{O}_{p}^{\prime}(E)=a^{\dagger} a-\frac{E^{2}}{\hbar^{2} g^{2}} e^{-6 r}+\frac{\Delta_{a}^{2}}{4 g^{2}} e^{-2 r}+1  \tag{A.12}\\
\hat{O}_{m}^{\prime}(E)=a^{\dagger} a-\frac{E^{2}}{\hbar^{2} g^{2}} e^{-6 r}+\frac{\Delta_{a}^{2}}{4 g^{2}} e^{-2 r} \tag{A.13}
\end{gather*}
$$

where we have multiplied through by a constant as these operators are used in $\hat{O}(E)\left|\psi_{E}^{-}\right\rangle=0$. From here we arrive at the eigenvalue equations in Eq. (3.43).

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